article

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Singular pure braid group SP_n

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Abstract. *This paper aims to prove some relationships of the singular pure braid group. Relations between singular braid group and singular pure braid group are presented in the case of 4 stranded braids.*

Key words. *Braid, Symmetric group, Singular braid monoid, Singular braid group, Singular pure braid group, Fundamental word.*

Sommario. *In questo lavoro si intende provare alcune relazioni valide al gruppo treccia singolare pura. Si presentano relazioni tra il gruppo treccia singolare e il gruppo treccia singolare pura concentrandosi al caso che essi hanno solo 4 filamenti.*

Parole chiave. *Treccia, Gruppo simmetrico, Monoide di treccia singolare, Gruppo treccia singolare, Gruppo treccia singolare pura, parola fondamentale.*

Introduction

Braids date back several centuries and have been used universally for ornamental purposes such as hats braids and Arab decorations; and for practical ones (the creation of ropes, fabrics, electric wires, metal tie rods, ...). Today they are described by means of abstract models grouped in the "theory of braids". Braid theory studies the concept of "braid", as in common imagination, and the multiple generalizations that originate in the various branches of mathematics. The idea that binds all the facets and aspects of the braids is that they, if appropriately structured, form a monoid or a group.

In a recent investigation (Ligouras [9]) some new properties of the singular braid monoid SB_n were highlighted. The main purpose of this article is to present and deepen some properties of the singular pure braid group SP_n and of the particular case of the SP_4 group which is formed by four filaments.

Preliminaries

SG_n group has a presentation made by the same set of generators and defining relators as the presentation of the *singular braid monoid* SB_n [11, 9], with the additional property of having its element invertible.

Let X be an arbitrary set. Every bijective application $f : X \to X$ is called a *permutation* of X. We denote by $S(X)$ the set of all the permutation of X.

$$S(X) := \{ f \mid f : X \to X,\ f \text{ bijective}\}.$$

Let ○ be the binary operation of the composition of applications:

$\circ : S(X) \times S(X) \to S(X)$, such that for every $f, g \in S(X)$ and for every $x \in X$ it results

$$(f \circ g)(x) \equiv fg(x) = f(g(x)).$$

Let $id: X \to X$ be the identity application of the set $S(X)$ for the operation $\circ$:
for every $f \in S(X)$ and for every $x \in X$ it results

$$(id \circ f)(x) = id(f(x)) = f(x) = f(id(x)) = (f \circ id)(x).$$

Proposition. The ordered triplet $(S(X), \circ, id)$ is a group.
The group $S(X)$ is called the *permutation group* of X.
If the set $X = \{x_1, x_2, ..., x_n\}$ is finite and of cardinality n ($|X| = n$), the notation can simplified by writing the set $I_n = X = \{1, 2, \dots, n\}$.
In this particular case (X is finite and of cardinality n) we write S_n instead of $S(X)$.
The group $(S_n, \circ, \mathrm{id})$ is called a *symmetric group* of degree n.
The elements of the symmetric group are one-to-one and onto applications.
It is emphasized that if $X = \{1, 2, \dots, n\}$, every element f of S_n, as a one-to-one correspondence of X in itself, represents a permutation of $\{1, 2, \dots, n\}$.
Let the set $X = \{1, ..., n\}$, the group $(S_n, \circ, id)$, m a positive integer, $2 \le m \le n$, and m distinct elements $i_1, i_2, ..., i_m \in X$. If a $\gamma \in S_n$ is such that:

1) $\gamma(i_k) = i_{k+1}$ se $1 \le k \le m-1$

2) $\gamma(i_k) = i_k$ se $i_k \notin \{i_1, i_2, ..., i_m\}$

3) $\gamma(i_m) = i_1$

γ is called a *cycle of length m*. The number m is called the *length of the cycle*.
To denote a cycle γ of length m we write $\gamma = (i_1, i_2, \dots, i_m)$.

A cycle γ of length m is also called an *m-cycle*.
A 2-*cycle* is also called *transposition* or *exchange*.
A 1-*cycle* only exists by convention and is (id).
The concept of transposition can also be defined as follows:
A transposition is a permutation that swaps only two objects and leaves the remainder fixed.
In this paper, the term transposition means those permutations that only exchange two consecutive elements which we indicate $s_i := (i, i+1)$ with $1 \le i \le n$ and which we call *simple transpositions* o *transpositions*.

Theorem ([8]). The group S_n has the following presentation:

$$S_n = \left\langle s_1,\ s_2, ..., s_{n-1} \;\middle|\; \begin{array}{ll} s_i^{\,2} = 1 & \text{for every } i \in \{1, ..., n-1\} \\ s_i s_j = s_j s_i & \text{for every } |i - j| > 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for every } i \in \{1, ..., n-2\} \end{array} \right\rangle.$$

Let a natural number $n \ge 1$. The *singular braid group* on n strands, denoted by SG_n, is the abstract group generated, in the complete version [6], by

- *real generators* $\sigma_1, \ldots, \sigma_{n-1}$ and their inverses $\sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}$
- *singular generators* $x_1, \ldots, x_{n-1}$ e $x_1^{-1}, \ldots, x_{n-1}^{-1}$

from the three B_n's group *real relations*

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j|>1$

(2) $\sigma_i\sigma_i^{-1} = 1_n = \sigma_i^{-1}\sigma_i$ for $i = 1, 2, \ldots, n-1$

(3) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \ldots, n-2$

from the *singular relations*

(4) $x_i x_j = x_j x_i$ for $|i-j|>1$

(5) $x_i x_i^{-1} = 1_n = x_i^{-1}x_i$ for $i = 1, 2, \ldots, n-1$

and, from the four *real-singular relations*

(6) $x_i\,\sigma_j = \sigma_j x_i$ for $|i-j|>1$

(7) $x_i\,\sigma_i = \sigma_i x_i$ for $i = 1, 2, \ldots, n-1$

(8) $x_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i x_{i+1}$ for $i = 1, 2, \ldots, n-2$

(9) $\sigma_i\sigma_{i+1}x_i = x_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \ldots, n-2$.

The following braid is called *fundamental word or Garside fundamental word* [2]:

$$\Delta := (\sigma_1\sigma_2 \ldots \sigma_{n-2}\sigma_{n-1})\,(\sigma_1\sigma_2 \ldots \sigma_{n-3}\sigma_{n-2}) \ldots (\sigma_1\sigma_2)\,\sigma_1.$$

To denote the fundamental word in B_n (Artin's braid group) we will use, according to some authors, the symbol Δ_n.

It is highlighted that for every $1 \le k \le n$ the following notation holds

$$\Delta_k := (\sigma_1\sigma_2 \ldots \sigma_{k-2}\sigma_{k-1})\,(\sigma_1\sigma_2 \ldots \sigma_{k-3}\sigma_{k-2}) \ldots (\sigma_1\sigma_2)\,\sigma_1.$$

The fundamental Garside braid Δ in B_n is inductively defined as:

$$\Delta_1 = 1,\ \Delta_2 = \sigma_1\Delta_1,\ \Delta_3 = \sigma_1\sigma_2\Delta_2,\ \ldots,\ \Delta \equiv \Delta_n = \sigma_1\sigma_2\ldots\sigma_{n-1}\Delta_{n-1} \text{ for every } n \ge 2.$$

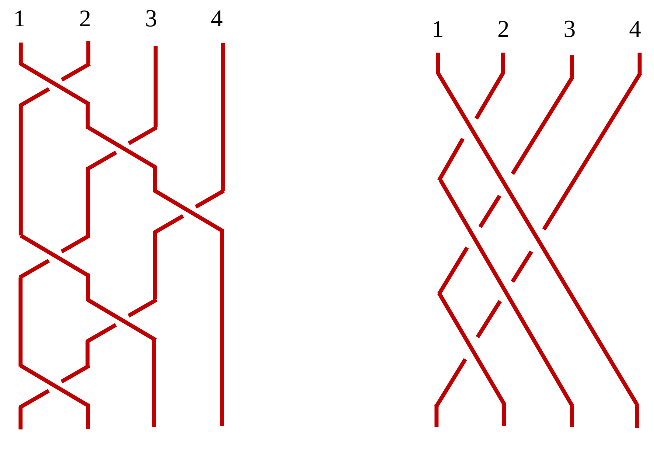

Fig. 1 – Different diagrams of the fundamental braid $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$ di SG_4

The figure (Fig.1) shows two equivalent diagrams of the fundamental word of the group SG_4.

To better understand this work and concepts of abstract algebra and braid groups, the author suggests [2, 3, 8, 10].

Some known activities of *SPn*

Lemma ([7]). Let SG_n be the singular braid group on n strands generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, x_1, x_2, \ldots, x_{n-1}$ and $s_1, s_2, \ldots, s_{n-1}$ the transpositions of the symmetric group S_n. Then a surjective homomorphism of groups exists

$$\nu_n : SG_n \to S_n$$

such that for every $1 \le i \le n-1$ results

$$\nu_n(\sigma_i) = s_i = (i, i+1) \quad \text{e} \quad \nu_n(x_i) = id.$$

Lemma ([7]). The following short exact sequence exists

$$\{id\} \longrightarrow ker(\nu_n) \longrightarrow SG_n \xrightarrow{\nu_n} S_n \longrightarrow \{id\}.$$

Lemma. Let a natural number $n > 1$, the group SG_n, the symmetric group S_n and the surjective homomorphism ν_n. Then the kernel, $ker(\nu_n)$, of ν_n, is a group.

Proposition ([7]). The group $ker(\nu_3)$ is isomorphic to the group SB_3.

Lemma ([1]). Let SG_n be the singular braid group n strands generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, x_1, x_2, \ldots, x_{n-1}$ and let $s_1, s_2, \ldots, s_{n-1}$ be the transpositions of the symmetric group S_n. Then a surjective homomorphism of groups exists

$$\upsilon_n : SG_n \to S_n$$

Such that, for every $1 \le i \le n-1$ results

$$\upsilon_n(\sigma_i) = \upsilon_n(x_i) = s_i = (i, i+1).$$

Lemma. Let a natural number $n > 1$, the group SG_n, the symmetric group S_n and the surjective homomorphism ν_n. Then the kernel, $ker(\nu_n)$, of ν_n, is a group.

The kernel of the surjective homomorphism υ_n is then called *singular pure braid group on n strands* and it is indicated with SP_n [5, 1]:

$$SP_n := ker(\upsilon_n : SG_n \to S_n).$$

Every element of the set SP_n is called *singular pure braid*.
A *pure singular braid diagram* is a diagram that represents geometrically an element of $\gamma \in SP_n$.

It is emphasized that the singular braids, although being a subset of the group of singular braids, are composed using the operation of vertical concatenation and that the identity element of their group, which is generally indicated with the symbol *id* or 1_n, is the braid with n vertical non-crossing strings.
Furthermore, it is clear that the permutation associated with a pure singular braid is the

permutation of identity in S_n.

Proposition ([1]). Let an integer number $n \geq 1$, SP_n the pure singular braid group on n strands, SG_n the singular braid group and S_n the symmetric group. Then

1) SP_n is a normal subgroup of SG_n:

$$SP_n \trianglelefteq SG_n.$$

2) the quotient group SG_n/SP_n is isomorphic to the group S_n:

$$SG_n / SP_n \cong S_n.$$

3) the index $(SG_n : SP_n)$ of subgroup SP_n in SG_n is equal to $n!$:

$$(SG_n : SP_n) = n!.$$

Proposition ([5]). The pure singular braid group SP_n contains a free abelian subgroup of rank $n - 1 + \lfloor n/2 \rfloor$.

Proposition ([5, 1]). The following short exact sequence holds

$$\{id\} \longrightarrow SP_n \longrightarrow SG_n \xrightarrow{\upsilon_n} S_n \longrightarrow \{id\}$$

Main results

The main result of this paper is the Theorem formulated later in this paragraph.

Lemma. If $\alpha = x_3x_2^{-1}x_1^{-1}x_2x_2x_3^{-1}x_2^{-1}x_1 \in SG_4$, then $\alpha^2 \in SP_4'$.

Proof. SP_4' denotes the derived group of SP_4. It is easy to see that the thread of the first braid α starting at position i_1 at the end is positioned at position f_3. The thread (or strand) with the starting position i_2 connects with the final position f_4, the thread starting at position i_3 ends at position f_1 and the thread starting at position i_4 connects with the position f_2. Similarly, the threads of the second braid α of the product have the following correspondences:
f_1 ends at ff_4, f_2 at ff_3, f_3 at ff_1, and f_4 at ff_2 thus the prove is complete. □

Corollary. Let the Garside element $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1 \in SG_4$ and $\alpha = \sigma_1^3\sigma_3^3\Delta^{-1} \in SG_4$. Then, $\alpha^2 \in SP_4'$.

Lemma. Let the elements $\alpha = x_3x_2^{-1}x_1^{-1}x_2x_2x_3^{-1}x_2^{-1}x_1 \in SG_4$ and $\beta \in SP_4$. Then, $\alpha\beta\alpha \in SP_4$.

Proof. Considering that the strands of the braid β start and end at the same position by definition, it allows us to follow the same procedure as the previous lemma to complete the prove. □

Corollary. Let the fundamental element $\Delta \in SG_4$ and the elements $\alpha = \sigma_1^3\sigma_3^3\Delta^{-1} \in SG_4$ and $\beta \in SP_4$. Then, $\alpha\beta\alpha \in SP_4$.

Theorem. Let an integer $k \geq 1$ and the elements $\gamma = x_3x_2^{-1}x_1^{-1}x_2^{2k}x_3^{-1}x_2^{-1}x_1 \in SG_4$ and $\beta \in SP_4$. Then, $\gamma\beta\gamma \in SP_4$.

Proof. Considering that the statement is valid for $k = 1$ as stated in the previous lemma and being well known that $x_2^{2(k-1)}$ is an element of SP_4, the ordered strands of the braids $x_3x_2^{-1}x_1^{-1}x_2^2$ and $x_3x_2^{-1}x_1^{-1}x_2^{2k}$ end at the same positions. This observation allows us to argue that $x_3x_2^{-1}x_1^{-1}x_2x_2x_3^{-1}x_2^{-1}x_1$ and γ end up at the same positions and the prove is complete. □

Corollary. Let an integer $k \geq 1$, the fundamental Garside's element $\Delta \in SG_4$ and the elements $\gamma = \sigma_1^{2k+1}\sigma_3^3\Delta^{-1} \in SG_4$ and $\beta \in SP_4$. Then, $\gamma\beta\gamma \in SP_4$.

Conclusion

In this paper, we presented the classical properties of the SP_n group and some of its links with other braid groups such as its derivative SP_n' group, SG_n and B_n groups, monoid SB_n and S_n symmetrical group. We then focused on the particular SP_4 group and presented some new relations both internal to the group and external with other groups.

In conclusion, I believe that the braid groups are suitable for further theoretical and applicative research, but also for didactic purposes, especially in universities and for high school students.

Declaration of Conflicting Interests

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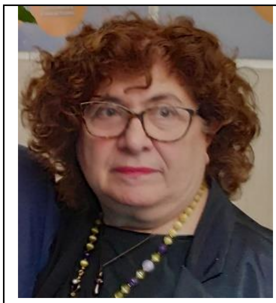

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