article

ISSN 2421-7247

The regular nonagon

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Abstract. *In the present work we study the construction of the regular nonagon, ie the division of the circle into nine equal parts. We prove that the construction with ruler and compass is impossible and we present the two regular star nonagons.*

Key words. *Normal nonagon, geometric construction, cyclotomy, regular star nonagon.*

Sommario. *Nel presente lavoro studiamo la costruzione dell'nonagono regolare, ovvero la divisione del cerchio in nove parti uguali. Dimostriamo che la sua costruzione con la riga e il compasso è impossibile e presentiamo i due nonagoni stellati regolari.*

Parole chiave. *Nonagono normale, costruzione geometrica, ciclotomia, regolare nonagono stelato.*

Introduction

The regular nonagon (or eneagon) is a convex polygon with nine equal sides and nine equal angles (see Fig. 1a). Every angle of the nonagon is 140°.

The regular nonagon is the second of the regular polygons, after the regular heptagon, that is not constructed by ruler and compass. The equilateral triangle, the square, the regular hexagon and octagon had been constructed by the ancient Greek Geometers.[1] About the regular heptagon we have the Archimedes' treatise "*On the regular Heptagon*" (*Περί του Κανονικού Επταγώνου*). In that treatise Archimedes constructs the regular heptagon by *neusis* (see EDIMAST, Volume 5, 2019, pp. 689-695). There is no reference in the ancient Greek literature about regular nonagon.

The central angle of the regular nonagon is equal to 40° (see Fig. 1a), so to construct it we must construct an angle of 40°. This could be constructed with ruler and compass if we could trisect an angle of 120°, which is the angle of the regular hexagon (Fig. 1b). Thus, the construction of the regular nonagon is associated with the problem of angle trisection. In fact, some historians of mathematics claim that the problem of angle trisection was posed by the attempt to be constructed the regular nonagon.[2]

In ancient Greek architecture we do not know buildings in the shape of a regular nonagon. The only relative known building is the *Philippeion* in Olympia, which was built by King Philip II (382-336 BC) and completed by his son Alexander the Great, who housed the statues of Philip's parents, his wife Olympias and Alexander. The building was surrounded by 18 Ionic columns on which a dome is rested. These columns were vertices of a regular octadecagon which is constructed by dividing the sides of a regular nonagon. Today only the base of this construction exists.[3]

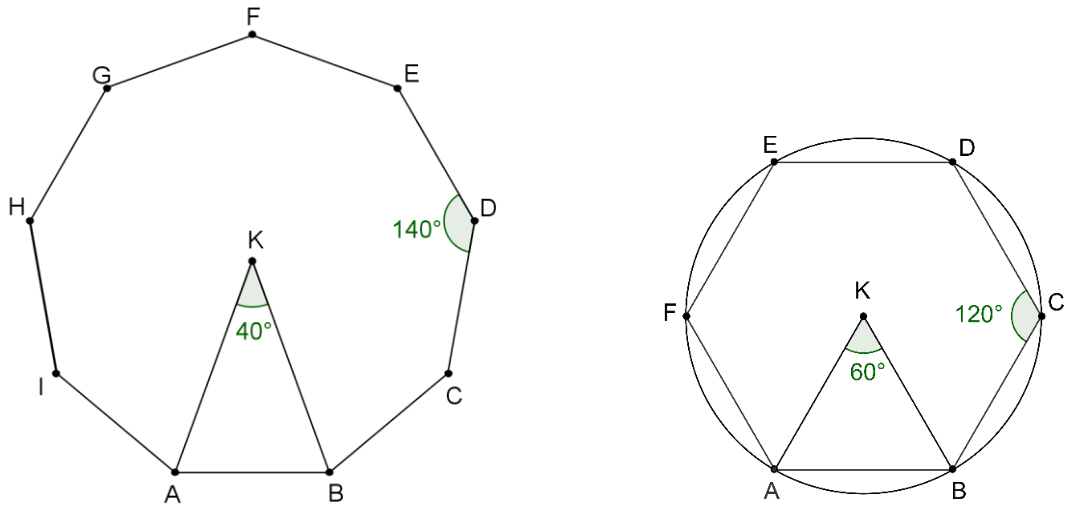

Fig.1 – The regular nonagon and hexagon

About dividing the circle into 9 equal parts

We will now negotiate the construction of the regular nonagon with the help of the complex plane. The method consists in calculating with the help of radicals the quantity $\cos\frac{2\pi}{9}$. $\theta=\frac{2\pi}{9}=40°$ is the central angle of the regular nonagon and $\cos\frac{2\pi}{9}=OA$ is the abscissa of the vertex P_1, as shown in figure (see Fig.2).

We consider at the complex plane the unit circle with center at the beginning of the axes. We also consider the equation

$$z^9-1=0.$$

The images of the roots of this equation in the complex plane are on the unit circle and divide it into 9 equal parts, ie they are the vertices of the regular nonagon.

The roots of the above equation in trigonometric form are given by the formula:

$$P_k=z_k=\cos\frac{2k\pi}{9}+i\cdot\sin\frac{2k\pi}{9},\ \text{for } k=0,1,2,\ldots,8.$$

The vertices of the regular nonagon P_k, $k = 0, 1, 2, \ldots, 8$ are shown in figure (see Fig.2). From the symmetry of the figure it follows that the values for the cosine of the roots z_1 and z_8 are equal, as for the cosines of the roots z_2 and z_7, z_3 and z_6, z_4 and z_5.

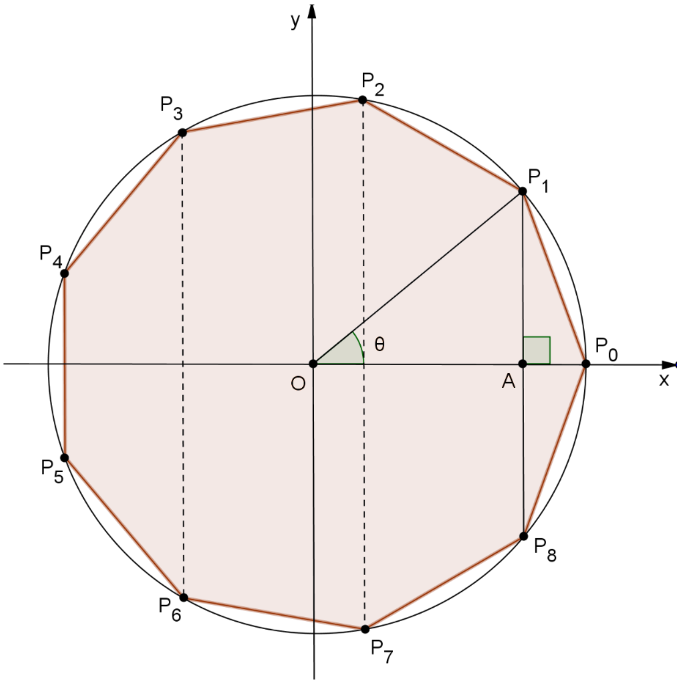

Fig.2 – The regular nonagon

If we set for the root $z_1 = x_1 + i\,y_1$, then for its cosine it will hold that:

$$x_1 = \mathrm{OA} = \cos\theta = x_8,\ \text{ where }\ \theta = \frac{2\pi}{9} = 40^\circ .$$

Similarly, for the cosines for the other roots we will have that:

$$x_2 = \cos 2\theta,\quad x_3 = \cos 3\theta,\quad x_4 = \cos 4\theta,\quad x_5 = \cos 5\theta,\quad x_6 = \cos 6\theta,$$

$$x_7 = \cos 7\theta \quad \text{and} \quad x_8 = \cos 8\theta,\ \text{ where }\ \theta = \frac{2\pi}{9} = 40^\circ .$$

We also have that:

$$x_2 = \cos 2\theta = x_7,\quad x_3 = \cos 3\theta = x_6 \quad \text{and} \quad x_4 = \cos 4\theta = x_5.$$

We now return to the equation $z^9 - 1 = 0$ (*) and have:

$$z^9 - 1 = 0 \iff (z-1)(z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0.$$

Since $z_0 = 1$ is its obvious root, we can divide the above equation by $z - 1$ and get the *cyclotomic equation* for the regular nonagon:

$$\frac{z^9 - 1}{z - 1} = 0 \iff z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0 \qquad (**),$$

which gives us the remaining six roots of equation (*), except for the obvious z = 1.

We have now

$$|z|=1 \;\Leftrightarrow\; |z|^2=1 \;\Leftrightarrow\; z\overline{z}=1 \;\Leftrightarrow\; \overline{z}=\frac{1}{z}.$$

We also set:

$$x = z+\frac{1}{z} = z+\overline{z} = 2\,\mathrm{Re}(z).$$

In the equation $z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$, since we have that $z \neq 0$, we can divide all the terms of the equation by z^4 and we will get:

$$z^4+z^3+z^2+z+1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}=0$$

$$\Leftrightarrow \quad (z^4+\frac{1}{z^4})+(z^3+\frac{1}{z^3})+(z^2+\frac{1}{z^2})+(z+\frac{1}{z})+1=0.$$

But we have that:

$$z^2+\frac{1}{z^2}=(z+\frac{1}{z})^2-2=x^2-2,$$

$$z^3+\frac{1}{z^3}=(z+\frac{1}{z})^3-3(z+\frac{1}{z})=x^3-3x$$

e

$$z^4+\frac{1}{z^4}=(z^2+\frac{1}{z^2})^2-2=\left(x^2-2\right)^2-2=x^4-4x^2+2.$$

So the equation is done

$$\left(x^4-4x^2+2\right)+\left(x^3-3x\right)+\left(x^2-2\right)+x+1=0$$

$$x^4+x^3-3x^2-2x-1=0 \qquad (***).$$

The roots of equation (***) are twice the abscissa of the vertices P_1, P_2, P_3, P_4 of the regular nonagon, since $x = 2\mathrm{Re}(z)$.

Equation (***) has root the –1, that is the root

$$x_3=\cos 3\theta=\cos 120°=-\frac{1}{2},$$

then with factorization we have: $x^4+x^3-3x^2-2x+1=0 \Leftrightarrow (x+1)(x^3-3x+1)=0$.

Finally, we have the equation: $x^3-3x+1=0$ **(****)**

The roots of equation (****) are twice the abscissa of the vertices P_1, P_2 and P_4 of the regular nonagon, since $x=2\,\mathrm{Re}(z)$.

Equation (***) is reductive and is 3rd degree (its degree is not a power of 2), so its roots according to the Wantzel's[4] criterion are not constructible by rule and compass.

Therefore, the normal nonagon is not constructible by ruler and compass.

Regular star nonagons

Definition

If a circle is divided into n equal arcs and we join the division points per m ($m < n$), then will be formed a regular zigzag line. If it is closed, then a *regular star polygon* is constructed.

Theorem

If a circle is divided into n arcs and we join the dividing points per m, $m < n$, a regular star polygon will be constructed with a number of sides n, if and only if the numbers m and n are prime to each other, ie if maximum common divisor $mcd(m, n) = 1$.

Proof

Each of the n equal arcs is equal to $\frac{1}{n}$ of the circle, so each chord subtended by an arc equal to $\frac{m}{n}$ of the circle. If α is the number of chords of the closed star zigzag line and k is the number of circles formed by the sum of the arcs then it will hold that:

$$\frac{m}{n}\cdot\alpha = k \quad \Leftrightarrow \quad \frac{m\cdot\alpha}{n} = k \qquad (*)$$

If the numbers m and *n are prime to each other then*, n must divide α, ie. α must be a multiple of n. Therefore, the smallest value of α is n, so $m = k$. So, we will have a regular star polygon with n sides if we cross the circle m times.

If now the numbers m and *n are not prime to each other*, then if we simplify the fraction $\frac{m}{n}$ with their maximum common divisor we will get the reducing fraction $\frac{m'}{n'}$. The formula (*) then is written $\frac{m'\cdot\alpha}{n'} = k$. Therefore if we divide the circle into n' equal parts, if we join the dividing points of the circle and if we cross the circle once we will get a regular star polygon with n' number of sides.

So, if we divide the circle into n equal arcs it is possible to construct a regular star polygon if we join the points of division per m, where m and n are prime each other.

Therefore, it is possible to construct so many regular star polygons with n sides, as are the numbers of the sequence 1, 2, ..., n–1, which are prime with n.

Nevertheless, regular star polygons with n sides that constructed if we join the dividing points of the circle per 2 or per n–2, per 3 or per n–3, etc. are identical.

Therefore the number of regular star polygons with number of sides n is equal to the number of integers 1, 2, ... , $\frac{n-1}{2}$ which are prime with n.

So, there are two regular star nonagons, the (9, 2) and (9, 4) as shown in the figure (Fig. 3), because $mcd(2, 9) = 1$ and $mcd(4, 9) = 1$, but $mcd(3,9) = 3 \neq 1$. The (9, 7) and (9, 5) are identified with (9, 2) and (9, 4) respectively.

The internal angles of the regular star nonagon (9, 2) are 100º, while of the (9, 4) are 20º (see Fig. 3). ■

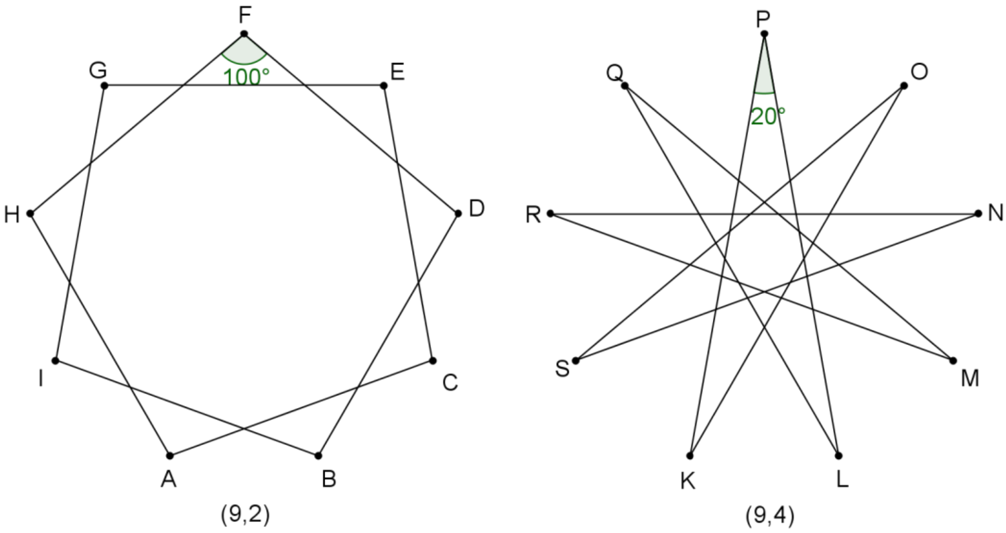

Fig.3 – The regular star nonagons

Mathematics is the queen of sciences and arithmetic the queen of mathematics.

C. F. Gauss

Note

1. E. Stamatis, *Euclid's Elements, Book IV*, Athens 1975.
2. T. L. Heath, *A History of Greek Mathematics* (translated in Greek), Athens 2001, p. 292.
3. D. Tsimbourakis, *Geometry in ancient Greece*, Athens 1985, p. 229.
4. Wantzel's criterion says that a number is constructible by ruler and compass if and only if is a root of a polynomial equation with coefficients rational numbers that is reductive and its degree is a power of 2. (See John Fraleigh, *A first course in Abstract Algebra*, Addison-Wesley, 1989, p.464).

Declaration of Conflicting Interests

The authors declared that they had no conflicts of interest with respect to their authorship or the publication of this article.

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Received September 12, 2020; *revised* October 14, 2020; *accepted* October 25, 2020; *published online* November 15, 2020