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article

The regular heptagon

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Abstract. *In the present work we study the construction of the regular heptagon, i.e. the division of the circle into seven equal parts. We prove that the construction with the ruler and compass is impossible and we give a construction with neusis attributed to Archimedes.*

Key words. *Normal heptagon, geometric construction, cyclotomy, neusis.*

Sommario. *Nel presente lavoro studiamo la costruzione dell'eptagono regolare, ovvero la divisione del cerchio in sette parti uguali. Dimostriamo che la costruzione con il regolo e il compasso è impossibile e diamo una costruzione con neusis attribuita ad Archimede.*

Parole chiave. *Eptagono normale, costruzione geometrica, ciclotomia, neusis.*

Introduction

The regular heptagon is the first of a series of regular polygons that is not constructed by rule and compass (see Stamatis (1961)) [1]. Prior to this, the isosceles triangle, the square, the regular pentagon and the regular hexagon had been constructed by the ancient Greek Geometers in order (see Sidiropoulos (1993)). It is therefore logical that the ancient Greek Geometers were engaged in the construction of the regular heptagon. For their efforts to construct it, we have no sources other than the treatise Archimedes' "*On the Normal Heptagon*" (*Περί του Κανονικού Επταγώνου*). This work has been saved in an Arabic copy of the Arabic mathematician Thabit ibn Qurra (826-901 AD) which was included in the work of the Persian mathematician and astronomer Al Birouni (962-1048 AD) "*Lessons in Trigonometry*" (see Gountouvàs (2017)). There we found Archimedes constructs the regular heptagon with a *neusis*[2], as we will see below.

The next non-constructible regular polygon with rule and compass is the regular enneagon with a central angle of 40°. It has been shown that the only integer angles that can be constructed by rule and compass are multiples of 3. Since the 40° angle is not a multiple of 3 it follows that the regular enneagon is not constructible. In the regular heptagon the central angle $\frac{2\pi}{7}=\frac{360^{\circ}}{7}$ is that it is not an integer, so we cannot decide on its constructability with the above criterion. We will see later with the help of the Wantzel criterion[3] why this is not buildable.

About dividing the circle into 7 equal parts

We will now negotiate the construction of the regular heptagon with the help of the complex plane. The method consists in calculating with the help of roots the quantity $\cos\dfrac{2\pi}{7}$. $\theta=\dfrac{2\pi}{7}$ is the central angle of the regular heptagon and $\cos\dfrac{2\pi}{7}$ is the abscissa *OA* of the vertex P_1, as shown in the figure below (Fig.1).

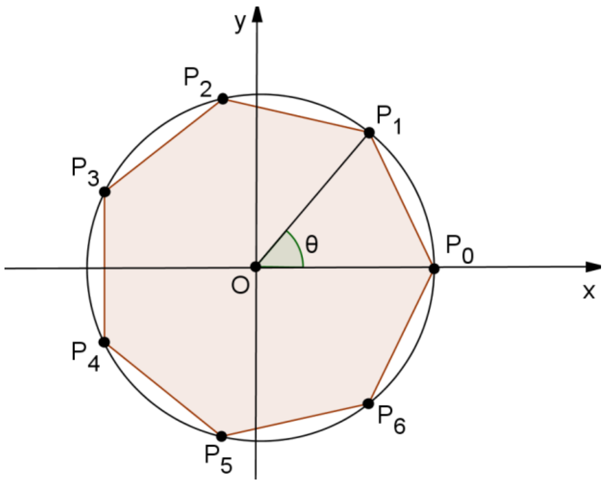

Fig.1 – The regular heptagon

We consider at the complex plane the unit circle with center at the beginning of the axes. We also consider the equation

$$z^7-1=0.$$

The images of the roots of this equation in the complex plane are on the unit circle and divide it into 7 equal parts, i.e. they are the vertices of the regular heptagon.

The roots of the above equation in trigonometric form are given by the relation:

$$P_k=z_k=\cos\frac{2k\pi}{7}+i\cdot\sin\frac{2k\pi}{7},\ \text{for } k=0,1,2,\ldots,6.$$

The vertices of the regular hexagon P_k, $k=0, 1, 2, \ldots, 6$ are shown in the figure above (see Fig.2). The symmetry of the figure shows that the values for the cosine of the roots z_1 and z_6 are equal, as for the cosine of the roots z_2 and z_5 and the roots z_3 and z_4.
If we set for the root $z_1=x_1+i\,y_1$, then for its cosine it will hold that:

$$x_1=\mathrm{OA}=\cos\theta=x_6,\ \text{where } \theta=\frac{2\pi}{7}.$$

Similarly, for the cosines of the other roots we will have that:

$$x_2=\cos2\theta,\ x_3=\cos3\theta,\ x_4=\cos4\theta,\ x_5=\cos5\theta,\ x_6=\cos6\theta,\ \text{ where } \theta=\frac{2\pi}{7}.$$

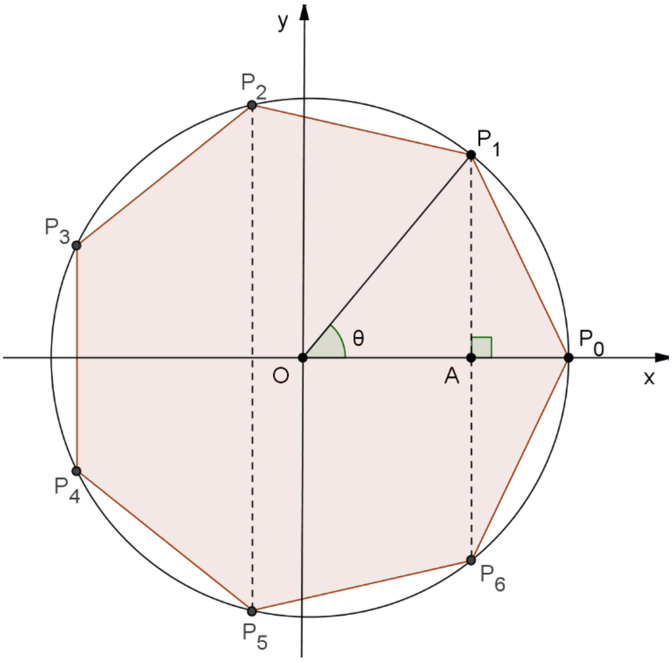

Fig.2 – The regular heptagon

We also have that:

$$x_2 = \cos 2\theta = x_5 \quad \text{and} \quad x_3 = \cos 3\theta = x_4.$$

We now return to the equation $z^7 - 1 = 0$ (*) and have:

$$z^7 - 1 = 0 \Leftrightarrow (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0.$$

Since $z_0 = 1$ is its obvious root, we can divide the above equation by $z - 1$ and get the *cyclotomic equation* for the regular heptagon:

$$\frac{z^7 - 1}{z - 1} = 0 \Leftrightarrow z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0 \quad (**),$$

which gives us the remaining six roots of equation (*), except for the obvious $z = 1$.

We have now

$$|z| = 1 \Leftrightarrow |z|^2 = 1 \Leftrightarrow z\bar{z} = 1 \Leftrightarrow \bar{z} = \frac{1}{z}.$$

We also set:

$$x = z + \frac{1}{z} = z + \bar{z} = 2\mathrm{Re}(z).$$

In the equation $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$, since we have that $z \neq 0$, we can divide all the terms of the equation by z^3 and we will get:

$$z^3+z^2+z+1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}=0 \Leftrightarrow (z^3+\frac{1}{z^3})+(z^2+\frac{1}{z^2})+(z+\frac{1}{z})+1=0 .$$

But we have that:

$$z^2+\frac{1}{z^2}=(z+\frac{1}{z})^2-2=x^2-2$$

$$z^3+\frac{1}{z^3}=(z+\frac{1}{z})^3-3(z+\frac{1}{z})=x^3-3x .$$

So the equation is done

$$x^3-3x+x^2-2+x+1=0$$

$$x^3+x^2-2x-1=0 \quad (***).$$

The roots of equation (***) are twice the vertices of the vertices P_1, P_2, P_3 of the regular heptagon, since x = 2Re(z).

Equation (***) is reductive and is 3rd degree (its degree is not a power of 2), so its roots according to the Wantzel criterion are not constructible by rule and compass.

Therefore, the normal heptagon is not buildable by rule and compass.

The Construction of Archimedes by neusis

Archimedes in his treatise "*Περί του Κανονικού Επταγώνου*", gives the construction of the regular heptagon in Proposition 17. In Proposition 16 he has given the construction with a straight line so that two parts meet a relation, with the help of which we make the second step of the construction.

Proposition 16.

Let ΑΒΓΔ be a square. Extend AB and make the diagonal ΑΓ. From point Δ we have a semicircle that intersects ΑΓ at T and the extension AB at Z so that the triangles BZE and ΓΔT are equivalent (isequal). From point T we bring the ΚΛ parallel to ΑΔ. Then it holds that $AB \cdot KA = BZ^2$ and $ZK \cdot BK = KA^2$. Also each of the sections BZ and AK is larger than BK.

The construction of the semi-straight ΔZ is done by neusis.

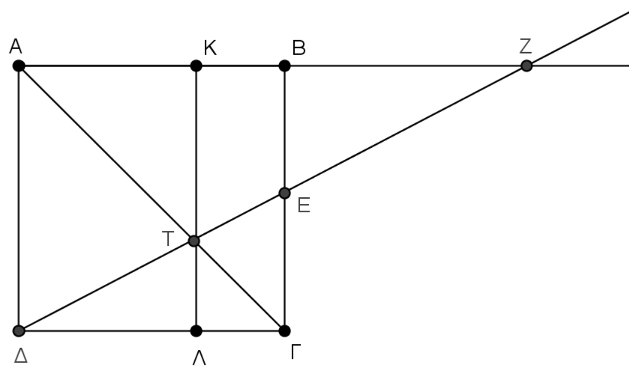

Fig. 3 – The regular heptagon

Proposition 17.

Construct a regular heptagon.

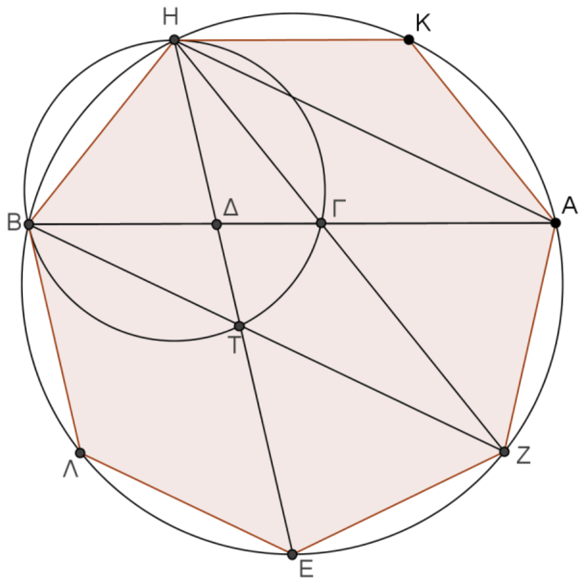

Fig. 4 – Construct a regular heptagon

Construction

1. The linear segment AB is considered as given.
2. On this we get two points Γ and Δ such that $A\Delta \cdot \Gamma\Delta = \Delta B^2$ and $\Gamma B \cdot \Delta B = A\Gamma^2$ (The construction is done with the help of Proposal 16).
3. We now construct the triangle ΓHΔ, so that ΓH = ΓA and ΔH = ΔB.
4. Construct the circumscribed circle in the triangle AHB.
5. Extend the segments HΓ and HΔ until they divide the circle at the points Z and E respectively.
6. The midpoint of the arc AH is determined and the vertex K is identified.
7. The midpoint of the arc BE is determined and the vertex Λ is identified.
8. The hexagon AKHBΛEZ is regular.

Proof

The triangle AΓH is isosceles and consequently the angles $B\hat{A}H$ and $A\hat{H}Z$ are congruent. Since $B\hat{A}H = A\hat{H}Z$ also the arcs BH and AZ are congruent.

The triangles AHΔ and ΓHΔ are similar because they have a common angle in Δ and

$$\frac{\Delta H}{A\Delta} = \frac{\Gamma\Delta}{\Delta H},$$

since $A\Delta \cdot \Gamma\Delta = \Delta B^2$ with ΔH = ΔB. So the arcs BH and ZE are equal.

The angles $\mathrm{T\hat{B}\Gamma = T\hat{H}\Gamma = \Gamma\hat{H}A}$ are equal since go to the arc AZ, so the points Γ, H, B, T are homocyclic. So the arcs BTΓ and HΓT are equal (sum of equal arcs), so BΓ = HT (*)

From the relation ΓB·ΔB = $\mathrm{A\Gamma^2}$ we have

$$\mathrm{\Gamma B\cdot \Delta B = A\Gamma^2 \Leftrightarrow HT\cdot \Delta H = \Gamma H^2 \Leftrightarrow \frac{HT}{\Gamma H} = \frac{\Gamma H}{\Delta H}},$$

We now have for the angles:

$$\mathrm{\Gamma\hat{T}\Delta = \Delta\hat{\Gamma}H},\ \mathrm{\Delta\hat{\Gamma}H = 2\cdot\Gamma\hat{A}H}\ \text{ and }\ \mathrm{\Gamma\hat{T}\Delta = \Delta\hat{B}H}$$

so $\mathrm{\Delta\hat{B}H = 2\cdot\Gamma\hat{A}H}$ then the arc AH is twice the BH and exactly the arc EB is twice BH.

Finally, since K and L are the means of the arcs AH and BE we have for the 7 arcs that

$$\overset{\frown}{\mathrm{AK}} = \overset{\frown}{\mathrm{KH}} = \overset{\frown}{\mathrm{HB}} = \overset{\frown}{\mathrm{B\Lambda}} = \overset{\frown}{\mathrm{\Lambda E}} = \overset{\frown}{\mathrm{EZ}} = \overset{\frown}{\mathrm{ZA}},$$

therefore

$$\mathrm{AK = KH = HB = B\Lambda = \Lambda E = EZ = ZA},$$

so the polygon AKHBΛEZ is a regular heptagon. ■

When people do not remember Aeschylus, because
languages are born and die, they will remember
Archimedes because mathematical concepts are eternal.

G. H. Hardy

Note

1. The regular heptagon is not constructible with *compass* and *ruler* but is constructible with a *marked ruler* and *compass*. This type of construction is called a *neusis construction*. The impossibility of straightedge and compass construction follows from the observation that $2\cos\frac{2\pi}{7} \approx 1.24$ is a zero of the irreducible cubic polynomial $x^3 + x^2 - 2x - 1$ which is the *minimum polynomial* of the quantity $2\cos\frac{2\pi}{7}$. It is well known that the degree of the minimum polynomial for a constructible number must be a power of 2.
2. The *neusis construction* consists of fitting a line element of given length l in between two given lines (r_1 and r_2), in such a way that the line element or its extension passes through a given point A. That is, one end of the line element l has to lie on r_1, the other end on r_2, while the line of element l passes through point A. Length l is called the *diastema*. Point A is called the *pole* of the neusis, line r_1 the *directrix* and line r_2 the *catch line*.
3. Let be an integer $n > 2$. A regular polygon with n sides can be constructed with ruler and compass if and only if n is the product of a power of 2 and any number of distinct Fermat primes. A *Fermat prime* is a prime number of the form $2^{\left(2^k\right)} + 1$.

Declaration of Conflicting Interests

The authors declared that they had no conflicts of interest with respect to their authorship or the publication of this article.

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