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article

Upper singular braid monoid SB_n^+

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Abstract. *The purpose of this paper is to describe the structure of SB_n^+ monoids and prove some new properties. The second purpose of the article is to present the experience carried out with mathematics teachers and students from 17 to 21 years old concerning SB_n^+ monoids and observe how these concepts were perceived.*

Key words. *Braid, Upper singular braid, Upper singular braid monoids.*

Sommario. (Monoide di treccia singolare positiva). *Lo scopo di questo articolo è quello di descrivere la struttura dei monoidi SB_n^+ e provare alcune nuove proprietà. Secondo scopo dell'articolo è presentare l'esperienza svolta con docenti di matematica e studenti da 17 a 21 anni di età inerente i monoidi SB_n^+ e osservare come questi concetti siano stati percepiti.*

Parole chiave. *Treccia, Treccia singolare positiva, Monoidi treccia singolare positiva.*

Introduction, Preliminaries and Notations

The main results of this paper are Theorem 1, Theorem 2 and Theorem 3.
Many researchers, since the last decades of the last century, have concentrated their research on the braid groups. From a mathematical point of view, everything began, in an organized way, with the first paper [1] by Emil Artin from 1925 and the second paper [2] from 1947. The braid groups of Artin is indicated with B_n. Since then many insights into the knowledge of the braid groups have been made. Simultaneously and subsequently the researchers began to explore different subgroups and generalizations of the B_n group. Generally, the definitions of the braid groups use the concepts of *generator* and *relations* to formalize their *presentations*.

Let be an integer $n \geq 1$. Recall that the complete *canonical presentation* [15] of the Artin braid group B_n on n strings involves $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ their opposite $\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_{n-1}^{-1}$ and the defining system of relations

(R0) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$

(R2) $\sigma_i\sigma_i^{-1} = 1_n = \sigma_i^{-1}\sigma_i$ for $i = 1, 2, \ldots, n-1$

(R3) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \ldots, n-2$.

The generators $\sigma_1, \ldots, \sigma_{n-1}$ are called *canonical generators* or *classical generators* [6]. The relations (R0) and (R3) are called canonical relations or classical relations. Each $\beta \in B_n$ element is called a *braid* or *n-strands braid*. Each letter of the alphabet of B_n is called a *strand*. Each of the generators σ_i, in a geometric meaning, represents the twisting of two adjacent strands around each other in a specific direction. Each relationship in the B_n group is a consequence of (R0) or (R2) or (R3) or two of them or all three relations of the definition of B_n.

While the $\beta \in B_n$ elements are called *braids* the elements of the group $\langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ generated by the $\sigma_1, \ldots, \sigma_{n-1}$ generators but without respecting the relationships of braid (R0), (R3) and (R2) are called *braid words*.

Consider *n* strands. Each strand has two ends an upper and a lower one. A braid diagram or braid diagram on *n*-strands consists of *n* strands with fixed upper end points and movable lower terminals. The upper endpoints of the threads are aligned from left to right on a horizontal (upper) line and the lower endpoints are aligned from left to right on a horizontal (lower) line. So having the lower ends free the strands can be intertwined. Each weave consists of the exchange of two successive lower end points, applying a half twist that can be carried out both clockwise and counter clockwise if viewed from above (see Fig. 1) this type of twist is called *elementary weaving movement* [12]. Braiding *n* strands consists in repeating a finite number of times the elementary intertwining movement in all *n* part of them. After finishing the weave and making the desired braid, the threads can still move individually, provided that their upper and lower ends remain motionless, and that the threads do not touch each other.

The interruption of the strand labelled as 2 that we notice in the figure (see Fig. 1) means that this strand passes under the strand labelled as 1.

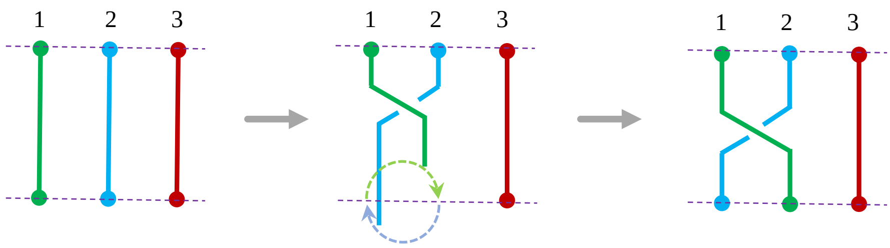

Fig. 1 – Elementary intertwining movement by applying a positive half twist in B_3.

The structure of the intersection is not real but it is an artefact of the projection on the plane.

The *positive canonical generators* $\sigma_1, \ldots, \sigma_{n-1}$, considered from a geometric point of view, represent the weaves made by applying a half twist in an anticlockwise direction [19 p.392, 12 p.36, 9], while the *negative canonical generators* $\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_{n-1}^{-1}$ represent the weaves made by applying a half twist in a clockwise direction.

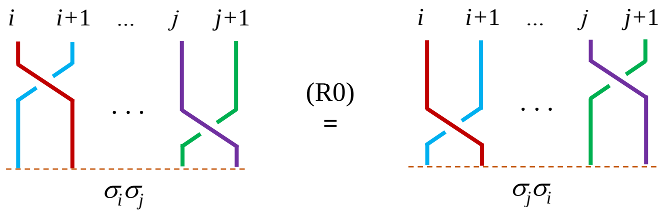

Fig. 2 – The relation (R0) of the definition of the braid group B_n

It can be thought that in a half twist in an anticlockwise direction the left filament passes from above from the right filament and in that point an appropriate generator is placed σ_i, while if the half twist is carried out clockwise the left filament passes under the right filament and corresponds to a suitable generator σ_i^{-1}. These two types of crosses are called *real crossings* or *classical crossings*.

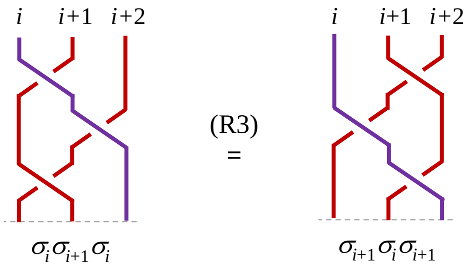

Fig. 3 – The relation (R3) of the definition of the braid group B_n

The figure (see Fig. 2) is a graphical representation of the relation (R0) while the figure (see Fig. 3) is a graphical representation of the relation (R3) of the B_n group.

The product between two braids β, $\gamma \in B_n$, with $n \geq 2$, is their *concatenation* (or *juxtaposition*) and generates a new braid that we call $\beta\gamma \in B_n$. The concatenation, which is a binary operation, is carried out in this way: the γ braid is positioned under (or on the right) the β braid so as to be able to join the end of the first strand of the β with the first of the γ and continue the same operation in sequence with the other strands of the two braids one after the other and end after connecting the end of the umpteenth strand of the β with the beginning of the umpteenth strand of the γ. The new braid that has formed is the $\beta\gamma$. The braid 1_n is the identity element of B_n. The figure (Fig. 4) presents this operation.

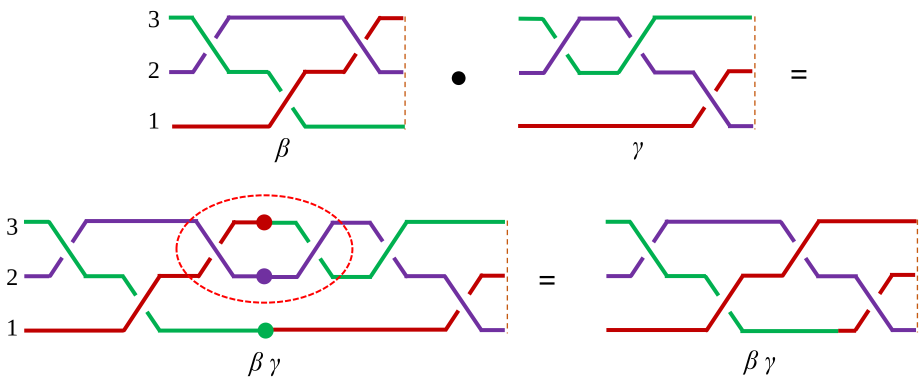

Fig. 4 – The $\beta\gamma$ product between two β and γ braids of the B_3 group

The *singular braid monoid* or *singular braid monoid on n-strands* or *Baez-Birman monoid* was introduced the same period and independently of Baez in 1992 [3] and by Birman in 1993 [5].

Let be an integer $n \geq 2$. The *singular braid monoid* or *Baez-Birman monoid*, indicated with SB_n, is the abstract monoid generated by the:

- *canonical generators* $\sigma_1, \ldots, \sigma_{n-1}$ and their inverse $\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_{n-1}^{-1}$,
- *singular generators* $x_1, \ldots, x_{n-1}$,

from the *canonical relations* of B_n

(R0) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$

(R3) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \ldots, n-2$,

from the *trivial relations* of B_n

(R2) $\sigma_i\sigma_i^{-1} = 1_n = \sigma_i^{-1}\sigma_i$ for $i = 1, 2, \ldots, n-1$,

from the *singular relations*

(S0) $x_i x_j = x_j x_i$ for $|i-j| > 1$

and from the *mixed real-singular relations*

(RS0) $x_i\sigma_j = \sigma_j x_i$ for $|i-j| > 1$

(RS2) $x_i\sigma_i = \sigma_i x_i$ for $i = 1, 2, \ldots, n-1$

(RS3) $x_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i x_{i+1}$ for $i = 1, 2, \ldots, n-2$

(RS4) $\sigma_i\sigma_{i+1}x_i = x_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \ldots, n-2$.

In geometric representations, σ_i corresponds to the generator of the classic braid group B_n and x_i is a new generator representing the intersection of strands i and i+1. That is, a singular geometric braid is equal to a geometric braid of the B_n group, except for the fact that simple intersections of one string with another which are called singular crossings are allowed. Each singular intersection is a double point. Singular crossings are represented by placing a small circle around the point where the two strands meet transversely.
In this group of braids, the x_i generators have no inverses.

Each element of the SB_n set is called a *singular braid*.

The monoid SB_n is an extension of the monoid of B_n.

The figure (see Fig. 5) is a graphical representation of the generators σ_i, σ_i^{-1} and x_i of monoid SB_n.

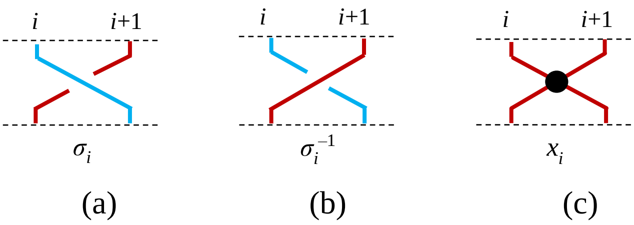

Fig. 5 – (a) the generator σ_i and (b) the generator σ_i^{-1} of B_n; (c) the generator x_i of SB_n

The monoid SB_n^+

The definition of SB_n^+ also uses generators and relationships as the building blocks.

Let be an integer $n \geq 2$. The *upper singular braid monoid* [6, 22], indicated with SB_n^+, is the abstract monoid generated by the:

- *canonical generators* $\sigma_1, \ldots, \sigma_{n-1}$,
- *singular generators* $x_1, \ldots, x_{n-1}$,

from the *canonical relations* of B_n

(R0) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| > 1$

(R3) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \ldots, n-2$,

from the *singular relations*

(S0) $x_i x_j = x_j x_i$ for $|i - j| > 1$

and from the *mixed real-singular relations*

(RS0) $x_i\sigma_j = \sigma_j x_i$ for $|i - j| \geq 2$

(RS2) $x_i\sigma_i = \sigma_i x_i$ for $i = 1, 2, \ldots, n-1$

(RS3) $x_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i x_{i+1}$ for $i = 1, 2, \ldots, n-2$

(RS4) $\sigma_i\sigma_{i+1}x_i = x_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \ldots, n-2$.

A *positive singular braid diagram* is a diagram that visually represents any $\beta \in SB_n^+$ on the plane. Each $\beta \in SB_n^+$ is called *upper singular braid*.

The alphabet of SB_n^+ is $\underline{A} = \{\sigma_1, \ldots, \sigma_{n-1}, x_1, \ldots, x_{n-1}\}$.

Two positive words X and Y on the alphabet $\underline{A}$ are *positively equal* or *positively equivalent* [6, 10] if they are equivalent. To indicate that X and Y are positively equal we use the symbol:

$$X \doteq Y.$$

To indicate that X and Y of SB_n^+ are *identical words* we use the symbol:

$$X \equiv Y.$$

Proposition 1 ([22]). Let be $i, j \in \{1, 2, \ldots, n-1\}$ and monoid SB_n^+.

Given the relation $\sigma_i A \doteq \sigma_j B$, where A and B are positive words in the alphabet $\underline{A}$, we have:

- if $i = j$, then $A \doteq B$;
- if $|i - j| = 1$, then $A \doteq \sigma_j\sigma_i C$, $B \doteq \sigma_i\sigma_j C$ for some word C;
- if $|i - j| \geq 2$, then $A \doteq \sigma_j C$, $B \doteq \sigma_i C$ for some word C.

Given the relation $\sigma_i A \doteq x_j B$, we have:

- if $|i - j| = 1$, then $A \doteq \sigma_j x_i C$, $B \doteq \sigma_i\sigma_j C$ for some word C;
- if $|i - j| \neq 1$, then $A \doteq x_j C$, $B \doteq \sigma_i C$ for some word C.

Given the relation $x_i A \doteq x_j B$, we have:

- if $j = 1$, then $A \doteq \sigma_j\sigma_i C$, $B \doteq \sigma_i\sigma_j C$ for some word C;
- if $|i - j| \geq 2$, then $A \doteq x_j C$, $B \doteq x_i C$ for some word C;

- if $|i - j| = 1$ impossible case.

A very important braid of the SB_n^+ monoid is the following [10]:

$$\Delta = \Delta_n := (\sigma_1\sigma_2 \ldots \sigma_{n-1})(\sigma_1\sigma_2 \ldots \sigma_{n-2})\ldots(\sigma_1\sigma_2)\sigma_1.$$

The braid Δ is called the *fundamental word* or the *fundamental word of Garside*.

The fundamental word Δ is inductively defined as follows:

$$\Delta_1 = 1, \quad \Delta_2 = \sigma_1\Delta_1, \quad \Delta_3 = \sigma_1\sigma_2\Delta_2, \quad \ldots, \quad \Delta_n = \sigma_1\sigma_2 \ldots \sigma_{n-1}\Delta_{n-1} \quad \text{for each } n \geq 2.$$

The *transformation of Garside words* $\mathcal{R}$ which is also called *reflection* in SB_n^+ or *flip* is defined by the rules [10]:

- $\mathcal{R}(\sigma_i) \equiv \sigma_{n-i}$
- $\mathcal{R}(x_i) \equiv x_{n-i}$.

Proposition 2 ([22]). Let them be the fundamental word Δ and the transformation of Garside words $\mathcal{R}$. In SB_n^+ the following relations are valid:

$$\sigma_i\Delta \doteq \Delta\mathcal{R}(\sigma_i) \quad \text{or equivalently} \quad \sigma_i\Delta \doteq \Delta\sigma_{n-i},$$

$$x_i\Delta \doteq \Delta\mathcal{R}(x_i) \quad \text{or equivalently} \quad x_i\Delta \doteq \Delta x_{n-i}.$$

The figure (see Fig. 6) is a graphical representation of the $x_1\Delta \doteq \Delta x_3$ relationship.

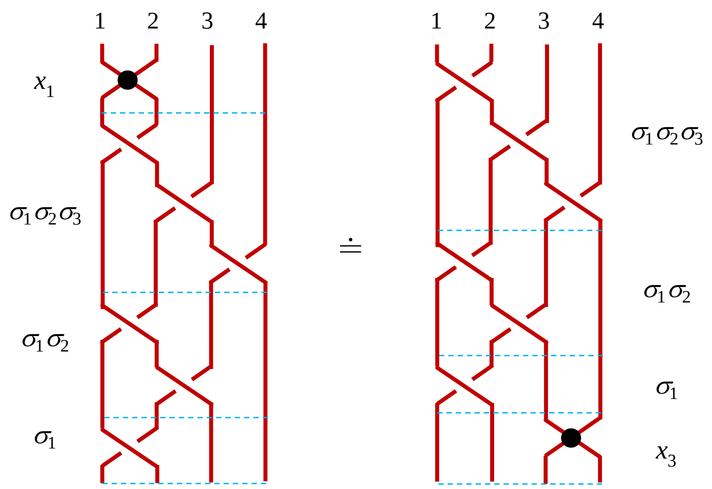

Fig. 6 – Equality $x_1\Delta \doteq \Delta x_3$ in SB_4^+

Proposition 3 ([22]). Let us be the monoids SB_n and SB_n^+. If

$$\pi : SB_n^+ \hookrightarrow SB_n$$

is *canonical homomorphism*, then π is a monomorphism.

The braid $\delta_{1,n}$

Theorem 1. Let be an integer $n \geq 3$, the monoid of upper singular braids SB_n^+ and the word

$$\delta_{1,n} = \sigma_{n-1}\dots\sigma_2\sigma_1^{\ 2}\sigma_2\dots\sigma_{n-1}.$$

If $\omega\in SB_n^+$ is a braid that contains only σ_i and x_i generators with $1 \leq i \leq n-2$, then we have

$$\omega\delta_{1,n} = \delta_{1,n}\omega.$$

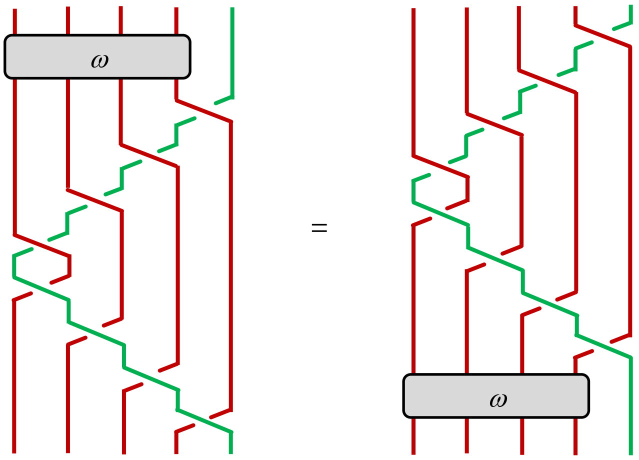

Fig. 7 – Graphical representation of the $\omega\delta_{1,5} = \delta_{1,5}\omega$ relation in SB_5^+

Proof. First we prove that for every $1 \leq i \leq n-2$ we have $\sigma_i\delta_{1,n} = \delta_{1,n}\sigma_i$.

$$\begin{aligned}
&\sigma_i\sigma_{n-1}\dots\sigma_2\sigma_1^{\ 2}\sigma_2\dots\sigma_{n-1}\\
&= \sigma_i\sigma_{n-1}\dots\sigma_{i+1}\sigma_i\dots\sigma_2\sigma_1^{\ 2}\sigma_2\dots\sigma_i\sigma_{i+1}\dots\sigma_{n-1}\\
&= \sigma_{n-1}\dots\sigma_i\sigma_{i+1}\sigma_i\dots\sigma_2\sigma_1^{\ 2}\sigma_2\dots\sigma_i\sigma_{i+1}\dots\sigma_{n-1}\\
&= \sigma_{n-1}\dots\sigma_{i+1}\sigma_i\sigma_{i+1}\dots\sigma_2\sigma_1^{\ 2}\sigma_2\dots\sigma_i\sigma_{i+1}\dots\sigma_{n-1}\\
&= \sigma_{n-1}\dots\sigma_{i+1}\sigma_i\dots\sigma_2\sigma_1^{\ 2}\sigma_2\dots\sigma_{i+1}\sigma_i\sigma_{i+1}\dots\sigma_{n-1}\\
&= \sigma_{n-1}\dots\sigma_2\sigma_1^{\ 2}\sigma_2\dots\sigma_{i+1}\sigma_i\sigma_{i+1}\dots\sigma_{n-1}\\
&= \sigma_{n-1}\dots\sigma_2\sigma_1^{\ 2}\sigma_2\dots\sigma_i\sigma_{i+1}\sigma_i\dots\sigma_{n-1}\\
&= \sigma_{n-1}\dots\sigma_2\sigma_1^{\ 2}\sigma_2\dots\sigma_i\sigma_{i+1}\dots\sigma_{n-1}\sigma_i\\
&= \sigma_{n-1}\dots\sigma_2\sigma_1^{\ 2}\sigma_2\dots\sigma_{n-1}\sigma_i.
\end{aligned}$$

For every $1 \leq i \leq n-2$ we have $x_i\delta_{1,n} = \delta_{1,n}x_i$. Indeed

$$\begin{aligned}
&x_i\sigma_{n-1}\dots\sigma_2\sigma_1^{\ 2}\sigma_2\dots\sigma_{n-1}\\
&= x_i\sigma_{n-1}\dots\sigma_{i+1}\sigma_i\dots\sigma_2\sigma_1^{\ 2}\sigma_2\dots\sigma_i\sigma_{i+1}\dots\sigma_{n-1}\\
&= \sigma_{n-1}\dots x_i\sigma_{i+1}\sigma_i\dots\sigma_2\sigma_1^{\ 2}\sigma_2\dots\sigma_i\sigma_{i+1}\dots\sigma_{n-1}\\
&= \sigma_{n-1}\dots\sigma_{i+1}\sigma_i x_{i+1}\dots\sigma_2\sigma_1^{\ 2}\sigma_2\dots\sigma_i\sigma_{i+1}\dots\sigma_{n-1}
\end{aligned}$$

$$= \sigma_{n-1}\ldots\sigma_{i+1}\sigma_i\ldots\sigma_2\sigma_1^{\ 2}\sigma_2\ldots x_{i+1}\sigma_i\sigma_{i+1}\ldots\sigma_{n-1}$$

$$= \sigma_{n-1}\ldots\sigma_{i+1}\sigma_i\ldots\sigma_2\sigma_1^{\ 2}\sigma_2\ldots\sigma_i\sigma_{i+1}x_i\ldots\sigma_{n-1}$$

$$= \sigma_{n-1}\ldots\sigma_2\sigma_1^{\ 2}\sigma_2\ldots\sigma_i\sigma_{i+1}\,x_i\ldots\sigma_{n-1}$$

$$= \sigma_{n-1}\ldots\sigma_2\sigma_1^{\ 2}\sigma_2\ldots\sigma_i\sigma_{i+1}\ldots\sigma_{n-1}x_i$$

$$= \sigma_{n-1}\ldots\sigma_2\sigma_1^{\ 2}\sigma_2\ldots\sigma_{n-1}x_i.$$

The braid ω is formed only by a combination of generators σ_i and x_i with $1 \leq i \leq n–2$. By repeating the procedure described above for each individual generator of ω in sequence and from left to right, the equality of the proposition is verified. □

In the figure (Fig. 6) it is noted that the strands 1, 2, ..., $n–1$ have no other interaction with the strand n beyond $\delta_{1,\,n}$ which represents its action. Consequently, both the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-2}, x_1, x_2, \ldots, x_{n-2}$ and the upper singular words formed by them, including ω, can move on the first $n–1$ strands without any interaction with strand n.

Corollary 1. Let be an integer $n \geq 3$, the monoid of the upper singular braids $SB_n^{\ +}$, two positive integers k, l such that $1 \leq k < l \leq n–1$ and the word

$$\delta_{k,\,l} = \sigma_{l-1}\ldots\sigma_{k+1}\sigma_k^{\ 2}\sigma_{k+1}\ldots\sigma_{l-1}.$$

If $\omega \in SB_n^{\ +}$ is a braid that does not contain generators σ_m and x_m, with $1 \leq m \leq k$ and $l–1 < m \leq n–1$, then

$$\omega\delta_{k,\,l} = \delta_{k,\,l}\,\omega.$$

The braid $\zeta_{1,\,n}$

Theorem 2. Let be an integer $n \geq 3$, the monoid of upper singular braids $SB_n^{\ +}$ and the word

$$\zeta_{1,\,n} = \sigma_{n-1}\ldots\sigma_2\,x_1^{\ 2}\sigma_2\ldots\sigma_{n-1}.$$

If $\omega \in SB_n^{\ +}$ is a braid that contains only σ_i and x_i generators with $2 \leq i \leq n–2$, then

$$\omega\zeta_{1,\,n} = \zeta_{1,\,n}\,\omega.$$

Proof. First we prove that for every $2 \leq i \leq n–2$ we have $\sigma_i\zeta_{1,\,n} = \zeta_{1,\,n}\sigma_i$.

$$\sigma_i\sigma_{n-1}\ldots\sigma_2\,x_1^{\ 2}\sigma_2\ldots\sigma_{n-1}$$

$$= \sigma_i\sigma_{n-1}\ldots\sigma_{i+1}\sigma_i\ \ldots\sigma_2\,x_1^{\ 2}\sigma_2\ldots\sigma_i\sigma_{i+1}\ldots\sigma_{n-1}$$

$$= \sigma_{n-1}\ldots\sigma_i\sigma_{i+1}\sigma_i\ldots\sigma_2\,x_1^{\ 2}\sigma_2\ldots\sigma_i\sigma_{i+1}\ldots\sigma_{n-1}$$

$$= \sigma_{n-1}\ldots\sigma_{i+1}\sigma_i\sigma_{i+1}\ldots\sigma_2\,x_1^{\ 2}\sigma_2\ldots\sigma_i\sigma_{i+1}\ldots\sigma_{n-1}$$

$$= \sigma_{n-1}\ldots\sigma_{i+1}\sigma_i\ldots\sigma_2\,x_1^{\ 2}\sigma_2\ldots\sigma_{i+1}\sigma_i\sigma_{i+1}\ldots\sigma_{n-1}$$

$$= \sigma_{n-1}\ldots\sigma_2\,x_1^{\ 2}\sigma_2\ldots\sigma_{i+1}\sigma_i\sigma_{i+1}\ldots\sigma_{n-1}$$

$$= \sigma_{n-1}\ldots\sigma_2\,x_1^{\ 2}\sigma_2\ldots\sigma_i\sigma_{i+1}\sigma_i\ldots\sigma_{n-1}$$

$$= \sigma_{n-1}\dots\sigma_2 x_1^2\sigma_2\dots\sigma_i\sigma_{i+1}\dots\sigma_{n-1}\sigma_i$$

$$= \sigma_{n-1}\dots\sigma_2 x_1^2\sigma_2\dots\sigma_{n-1}\sigma_i.$$

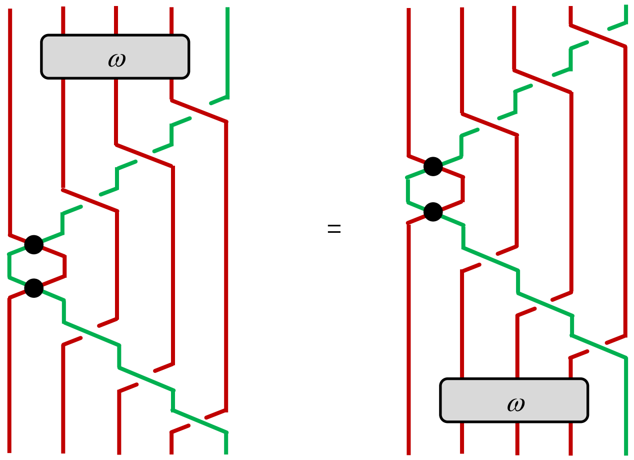

Fig. 8 – Graphical representation of the $\omega\zeta_{1,5} = \zeta_{1,5}\omega$ relation in SB_5^+

For every $2 \le i \le n-2$ we have $x_i\zeta_{1,n} = \zeta_{1,n}x_i$. Indeed,

$$x_i\sigma_{n-1}\dots\sigma_2 x_1^2\sigma_2\dots\sigma_{n-1}$$

$$= x_i\sigma_{n-1}\dots\sigma_{i+1}\sigma_i\dots\sigma_2 x_1^2\sigma_2\dots\sigma_i\sigma_{i+1}\dots\sigma_{n-1}$$

$$= \sigma_{n-1}\dots x_i\sigma_{i+1}\sigma_i\dots\sigma_2 x_1^2\sigma_2\dots\sigma_i\sigma_{i+1}\dots\sigma_{n-1}$$

$$= \sigma_{n-1}\dots\sigma_{i+1}\sigma_i x_{i+1}\dots\sigma_2 x_1^2\sigma_2\dots\sigma_i\sigma_{i+1}\dots\sigma_{n-1}$$

$$= \sigma_{n-1}\dots\sigma_{i+1}\sigma_i\dots\sigma_2 x_1^2\sigma_2\dots x_{i+1}\sigma_i\sigma_{i+1}\dots\sigma_{n-1}$$

$$= \sigma_{n-1}\dots\sigma_{i+1}\sigma_i\dots\sigma_2 x_1^2\sigma_2\dots\sigma_i\sigma_{i+1}x_i\dots\sigma_{n-1}$$

$$= \sigma_{n-1}\dots\sigma_2 x_1^2\sigma_2\dots\sigma_i\sigma_{i+1}x_i\dots\sigma_{n-1}$$

$$= \sigma_{n-1}\dots\sigma_2 x_1^2\sigma_2\dots\sigma_i\sigma_{i+1}\dots\sigma_{n-1}x_i$$

$$= \sigma_{n-1}\dots\sigma_2 x_1^2\sigma_2\dots\sigma_{n-1}x_i.$$

The braid ω is formed only by a combination of generators σ_i and x_i with $2 \le i \le n-2$. By repeating the procedure described above for each individual generator of ω in sequence and from left to right, the equality of the proposition is verified. □

Corollary 2. Let be an integer $n \ge 3$, the monoid of the upper singular braids SB_n^+, two positive integers k, l such that $1 \le k < l \le n-1$ and the word

$$\zeta_{k,l} = \sigma_{l-1}\dots\sigma_{k+1}x_k^2\sigma_{k+1}\dots\sigma_{l-1}.$$

If $\omega\in SB_n^+$ is a braid that does not contain generators σ_m and x_m, with $1 \le m \le k$ and $l-1 < m \le n-1$, then

$$\omega\zeta_{k,l} = \zeta_{k,l}\omega.$$

The braid $\xi_{1,n}$

Theorem 3. Let be an integer $n \geq 3$, the monoid of upper singular braids SB_n^+ and the word

$$\xi_{1,n} = \sigma_{n-1} \cdots \sigma_2 x_1 \sigma_1 \sigma_2 \cdots \sigma_{n-1}.$$

If $\omega \in SB_n^+$ is a braid that contains only σ_i and x_i generators with $2 \leq i \leq n-2$, then

$$\omega \xi_{1,n} = \xi_{1,n} \omega.$$

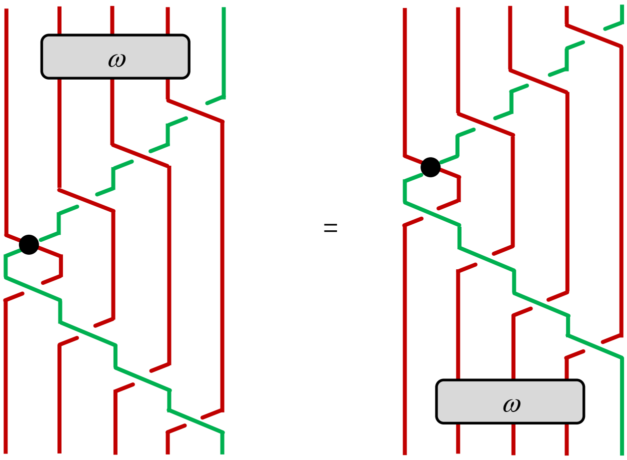

Fig. 9 – Graphical representation of the $\omega\xi_{1,5} = \xi_{1,5}\omega$ relation in SB_5^+

Proof. It is proved that for every $1 \leq i \leq n-2$ results $\sigma_i \xi_{1,n} = \xi_{1,n} \sigma_i$.

$$\begin{aligned}
&\sigma_i \sigma_{n-1} \cdots \sigma_2 x_1 \sigma_1 \sigma_2 \cdots \sigma_{n-1} \\
&\quad = \sigma_i \sigma_{n-1} \cdots \sigma_{i+1} \sigma_i \cdots \sigma_2 x_1 \sigma_1 \sigma_2 \cdots \sigma_i \sigma_{i+1} \cdots \sigma_{n-1} \\
&\quad = \sigma_{n-1} \cdots \sigma_i \sigma_{i+1} \sigma_i \cdots \sigma_2 x_1 \sigma_1 \sigma_2 \cdots \sigma_i \sigma_{i+1} \cdots \sigma_{n-1} \\
&\quad = \sigma_{n-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1} \cdots \sigma_2 x_1 \sigma_1 \sigma_2 \cdots \sigma_i \sigma_{i+1} \cdots \sigma_{n-1} \\
&\quad = \sigma_{n-1} \cdots \sigma_{i+1} \sigma_i \cdots \sigma_2 x_1 \sigma_1 \sigma_2 \cdots \sigma_{i+1} \sigma_i \sigma_{i+1} \cdots \sigma_{n-1} \\
&\quad = \sigma_{n-1} \cdots \sigma_2 x_1 \sigma_1 \sigma_2 \cdots \sigma_{i+1} \sigma_i \sigma_{i+1} \cdots \sigma_{n-1} \\
&\quad = \sigma_{n-1} \cdots \sigma_2 x_1 \sigma_1 \sigma_2 \cdots \sigma_i \sigma_{i+1} \sigma_i \cdots \sigma_{n-1} \\
&\quad = \sigma_{n-1} \cdots \sigma_2 x_1 \sigma_1 \sigma_2 \cdots \sigma_i \sigma_{i+1} \cdots \sigma_{n-1} \sigma_i \\
&\quad = \sigma_{n-1} \cdots \sigma_2 x_1 \sigma_1 \sigma_2 \cdots \sigma_{n-1} \sigma_i.
\end{aligned}$$

For every $2 \leq i \leq n-2$ we have $x_i \xi_{1,n} = \xi_{1,n} x_i$. Indeed,

$$\begin{aligned}
&x_i \sigma_{n-1} \cdots \sigma_2 x_1 \sigma_1 \sigma_2 \cdots \sigma_{n-1} \\
&\quad = x_i \sigma_{n-1} \cdots \sigma_{i+1} \sigma_i \cdots \sigma_2 x_1 \sigma_1 \sigma_2 \cdots \sigma_i \sigma_{i+1} \cdots \sigma_{n-1} \\
&\quad = \sigma_{n-1} \cdots x_i \sigma_{i+1} \sigma_i \cdots \sigma_2 x_1 \sigma_1 \sigma_2 \cdots \sigma_i \sigma_{i+1} \cdots \sigma_{n-1} \\
&\quad = \sigma_{n-1} \cdots \sigma_{i+1} \sigma_i x_{i+1} \cdots \sigma_2 x_1 \sigma_1 \sigma_2 \cdots \sigma_i \sigma_{i+1} \cdots \sigma_{n-1}
\end{aligned}$$

$$= \sigma_{n-1}\ldots\sigma_{i+1}\sigma_i\ldots\sigma_2 x_1\sigma_1\sigma_2\ldots x_{i+1}\sigma_i\sigma_{i+1}\ldots\sigma_{n-1}$$
$$= \sigma_{n-1}\ldots\sigma_{i+1}\sigma_i\ldots\sigma_2 x_1\sigma_1\sigma_2\ldots\sigma_i\sigma_{i+1}x_i\ldots\sigma_{n-1}$$
$$= \sigma_{n-1}\ldots\sigma_2 x_1\sigma_1\sigma_2\ldots\sigma_i\sigma_{i+1}x_i\ldots\sigma_{n-1}$$
$$= \sigma_{n-1}\ldots\sigma_2 x_1\sigma_1\sigma_2\ldots\sigma_i\sigma_{i+1}\ldots\sigma_{n-1}x_i$$
$$= \sigma_{n-1}\ldots\sigma_2 x_1\sigma_1\sigma_2\ldots\sigma_{n-1}x_i.$$

The braid ω is formed only by a combination of generators σ_i and x_i with $2 \le i \le n-2$. By repeating the procedure described above for each individual generator of ω in sequence and from left to right, the equality of the proposition is verified. □

Corollary 3. Let be an integer $n \ge 3$, the monoid of the upper singular braids SB_n^+, two positive integers k, l such that $1 \le k < l \le n-1$ and the word

$$\xi_{k,l} = \sigma_{l-1}\ldots\sigma_{k+1}x_k\sigma_k\sigma_{k+1}\ldots\sigma_{l-1}.$$

If $\omega \in SB_n^+$ is a braid that does not contain generators σ_m and x_m, with $1 \le m \le k$ and $l-1 < m \le n-1$, then

$$\omega\xi_{k,l} = \xi_{k,l}\omega.$$

Discussion and conclusions

Some of the contents of this contribution have been discussed with a group of teachers and students. The experiment lasted 8 hours and is the continuation of two other experiences carried out with the same students and related topics of abstract algebra [14] and combinatorics.

During the meetings few demonstrations of the propositions were proposed but many easy examples were presented and carried out. The trainees in collaborative mode, both during the meetings and between one meeting and another, have done some simple exercises.

After the obvious difficulties due to the initial impact with the new topic, the participants, managed to follow the activities with greater autonomy than the previous courses. Since the number of participants is limited, no generalized statements can be made.

The experience is continuing with new meetings, involving the same participants further deepening the groups of braids that lend themselves well as object for educational experiments at school in extra-school hours with pupils of 17-19 years old and at university with students of the first two years of scientific courses.

To study the concepts referred to during this experience, the students also consulted the following texts [7, 11, 16, 17, 18, 20, 21].

Declaration of Conflicting Interests

The authors declared that they had no conflicts of interest with respect to their authorship or the publication of this article.

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