article

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Multıple representatıons and development of students' self-confidence on rational number

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Abstract. *In spite of the fact that analytical programs change and mathematical school texts adapt to new education needs, students, internationally, continue to have difficulties when handling fractions. This paper presents the results of a research conducted on students of the 5th and 6th grade of elementary school and its purpose was to investigate at what extent the use of multiple representations is possible to help students cope with difficulties on mathematics and thus to boost their self-confidence in mathematics. For that purpose, the concepts of fractions was chosen are classification of fractions as a representation on the number line, as well as the concepts of the unit's division in equal parts and the concept of the improper fractions. Thus, we present education practices were applied by our research team. These teaching practices take into account the results as stated by international bibliographies as well as years of research of our team on rational numbers. They emphasize on multiple representations, use of experiential activities and activities carried out on electronic platforms. In additional, the present research deepens with semi-structured interviews of the participants. The results of the research indicate that students after instructive interventions with the use of multiple representations performed better on fractions and increased their self-esteem in mathematics.*

Key words. *Fractions, instructional practices, multiple representations, students' self-confidence.*

Sommario. *Nonostante il fatto che i programmi analitici cambino e che i testi scolastici matematici si adattino ai nuovi bisogni educativi, gli studenti, a livello internazionale, continuano ad avere difficoltà nel maneggiare le frazioni. Questo documento presenta i risultati di una ricerca condotta su studenti del quinto e sesto grado della scuola elementare e il suo scopo era quello di indagare in che misura l'uso di rappresentazioni multiple è possibile per aiutare gli studenti a far fronte alle difficoltà della matematica. A tale scopo, sono stati scelti i concetti di frazioni come classificazione delle frazioni come rappresentazione sulla linea numerica, nonché i concetti della divisione dell'unità in parti uguali e il concetto delle frazioni improprie. Quindi, presentiamo le pratiche educative applicate dal team di ricerca. Queste pratiche di insegnamento tengono conto dei risultati dichiarati dalle bibliografie internazionali e di anni di ricerca del nostro team su numeri razionali. Sottolineano su molteplici rappresentazioni, l'uso di attività esperienziali e attività svolte su piattaforme elettroniche. In aggiunta, la ricerca attuale si approfondisce con interviste semi-strutturate dei partecipanti. I risultati della ricerca indicano che gli studenti dopo interventi istruttivi con l'uso di rappresentazioni multiple hanno ottenuto risultati migliori sulle frazioni e aumentato la loro autostima in matematica.*

Parole chiave. *Frazioni, pratiche didattiche, rappresentazioni multiple, fiducia in sé stessi.*

Introduction

The purpose of this paper is to provide instructional practices over time implemented by the research team and its positive effect on fraction understanding and students' self-confidence in mathematics. These instructional practices consider the difficulties of students emerged both from the literature and years of research team on the rational numbers. They emphasise multiple representations and they made use experiential activities and digital representations. The content and activities of teachings were determined by the findings of timeless researches conducted since 2011 to this day and help students reduce the difficulties they face with the concept of fractions classification as a representation on the geometric model of number line, as well as with the concepts of unit division in equal parts and improper fractions with the help of multiple representations with the ultimate goal of developing students' self-confidence in mathematics.

These concepts were chosen because several researchers claim that they are essential for developing rational number meaning and also they are associated with the understanding of other mathematical meanings (Jordan et al., 2013). Lee and Shin (2015) indicate that the distributive partitioning operation was revealed in various mathematical problem situations such as fraction multiplication, fraction division, and multiplicative transformation between fractional quantities. Additionally, the knowledge of improper fractions associates with problem posing (Avgerinos, Vlachou, 2013). Moreover, the implicit use of fractional can lead to more explicit use of structures and relationships in algebraic situations (Empson et al. 2011; Hackenberg, 2013). The fractional knowledge, that is, influenced how students wrote equations to represent multiplicative relationships between two unknown quantities (Lee, Hackenberg, 2014). Other researchers claimed that the division of a unit into equal parts, is essential for developing rational number meaning (Kieren, 1992; Mack, 2001; Steffe, Olive, 2010) and for developing whole number understandings (Boyce, Norton, 2016).

Literature review

Multiple representations in mathematics

In education at least in some instances no understanding can be achieved without the aid of representation. Such a case is the notion of fractions. In mathematics education, the concept of representation is used as equivalent to a sign that shows and makes present a mathematical concept - a symbol or mark to think about the concept. Representations are those schemes or mental images with which the subjects work on mathematical ideas (Castro-Rodríguez et al., 2016). Particularly, it is usual to consider the duality, external and internal representations. To think about and to communicate mathematical ideas we need to represent them in some way. Communication requires that the representations be external, taking the variety of forms, including pictures (e.g., drawing, charts, graphs), written symbols (e.g., numbers, equations, words), manipulative models, oral language (e.g., talk between pairs of students and whole class discussion), and real-world situations (Ryken, 2009). The multiple representations are the use variety of these external representations during teaching a mathematics concept.

According to some researches (Card et al., 1999; Cuoco, Curcio 2001; Cheng, 2002; Dreher, Kuntze, 2015) representing mathematical objects in multiple ways plays an important role in mathematical understanding and brings value to teaching processes. In addition, recent trends in curriculum standards, including standards developed by the National Council of Teachers of Mathematics (NCTM, 2000), have highlighted the productive role that drawn models and other external representations can play in teaching and learning mathematics (Jacobson, Izsak, 2015). Although the representations add complexity, using a range of representations is necessary for developing children's fractions understanding because each provide links to the underlying fractions concepts and children require support to make active connections within and between the various representations (Hansen et al., 2016).

However, in 1993 Duval called attention to a cognitive paradox hidden within in various representations. Handling these representations choosing he distinguishing features of the concept we must treat and convert, is not learnt automatically. This learning results from a process of explicit teaching in which the teacher must render the student co-responsible. Teachers often underestimate this aspect and passing from one register to another, believing that the student follows. The teacher is able to jump from one register to another without problems, because he has already conceptualised: while in fact the student does not so, the student follows at the level of semiotic representatives, but not of meanings (Fandiño Pinilla, 2007).

Understanding and self-confidence in mathematics

The importance of beliefs in mathematics education is in concordance with the constructivist understanding of teaching and learning. We understand beliefs as "an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behaviour" (Schoenfeld, 1992:358). Mathematical beliefs can be divided into four main components: beliefs on mathematics, beliefs on oneself as a mathematics learner/applier, beliefs on teaching mathematics, and beliefs on learning mathematics (Lester et al., 1989). In this paper, we will deal with beliefs on oneself as a mathematics learner/applier, in other words, with self-confidence.

Several studies have shown that the learning of mathematics is influenced by a pupil's mathematics-related beliefs, especially self-confidence (Hannula & Malmivuori, 1996; House, 2000). So, self-confidence has a remarkable connection with success in mathematics. However, self-confidence and understanding self-esteem relate to the way of teaching. In other words, the passage from "Knowledge" (academic) to "learned knowledge" (of the student) is the result of a long and delicate path leading first to the knowledge to be taught, then to the knowledge actually taught and finally to the knowledge learnt. In this sequence, the first step of transforming "Knowledge" into "knowledge to teach" is called didactic transposition and constitutes a moment of great importance in which the professionalism and creativity of the teacher are of utmost importance (Fandiño Pinilla, 2007).

Teaching with representations

Research on international literature took place, reviewing current literature about representations in fractions in order to study and record the results of the surveys that have been published on the representations in fractions. In other words, which representations have emerged through these surveys as the most appropriate or inappropriate ones for the students to understand the notion of fractions and specifically, the concepts of classification of fractions as a representation on the number line, as well as the concepts of the unit's division in equal parts and the concept of the improper fractions. The review showed that there are hardly any teaching approaches and proposals made by the researchers on how teachers can deal with teaching and not only on the difficulties faced by students in fractions.

More specifically, as far as the number line is concerned, Brousseau et al. (2007) conducted a series of interventions in order to lead students' day by day to invent, understand and become very good at all aspects of both basic mathematical structures, the rational and decimal numbers. The intervention included a total of 65 courses (15 cycles) which were held in the fourth grade of Michelet school. The courses were repeated in two parallel classes with different teachers in a period of over 15 years, which means that more than 750 students have taken part in them. In the third lesson of the fifth cycle a representation of fractions on the number line takes place, leading gradually to the representation of the number line (see Fig. 1, 2, 3, 4) through a range of playful procedures and teaching methodology. According to this research, at the end of this activity, most students can quickly and undoubtedly put decimal fractions on the number line, and all students can analyze a decimal fraction in units, tenths, hundredths etc.

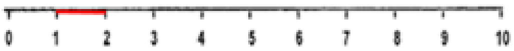

Fig. 1 – Interval [1,2)

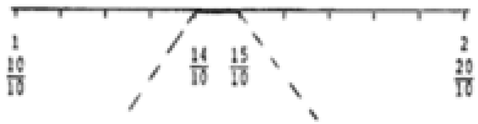

Fig. 2 – 14/10 and 15/10 on number line

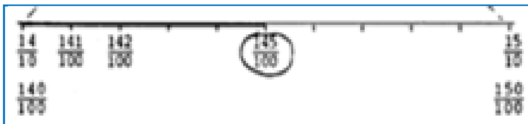

Fig. 3 – Number line points in hundredths

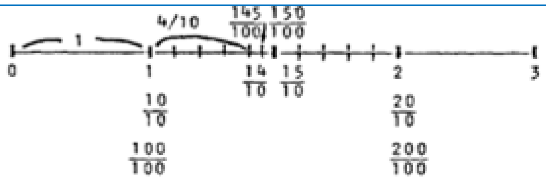

Fig. 4 – Finale form of number line

Another research proposal on the number line is that of Sedig and Sumner (2006) who reported the importance of visual representations of mathematics and the use of digital tools that enable them. One of these tools is the use of zoom on the number line (see Fig. 5). Zooming raises or lowers the level of detail on the number line, allowing students to visualise dividing numbers into equal parts and thus facilitate the transition to the field of rational numbers.

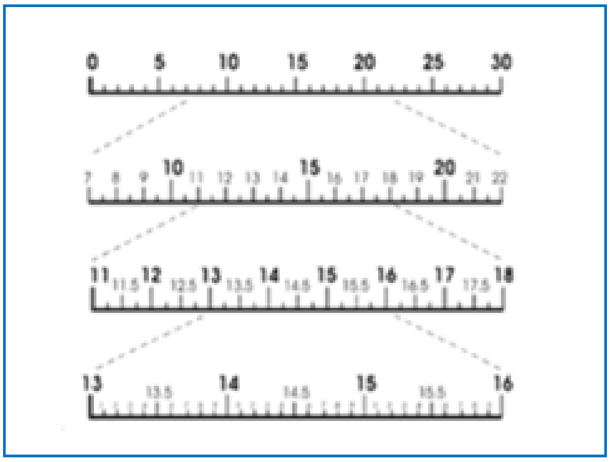

Fig. 5 - Several levels of conceptual zooming of a number line

Hansen et al. (2016) report their article the iTalk2Learn. ITalk2Learn is a Fraction Lab with the aim of developing an open-source intelligent tutoring platform that supports math's learning for students aged 5 to 11. It allows students to learn from a system in a more natural way than ever before. This empowers educators to deliver the right lesson at the right time for every child, enabling personalized learning at scale. In addition, this Fractions Lab utilized a variety of fractions representations including continuous and discrete fractions and fractions in one, two and three dimensions (number line, area/region, and liquid measures, respectively), developing, in this way, children's conceptual understanding with a virtual manipulative (see Fig. 6).

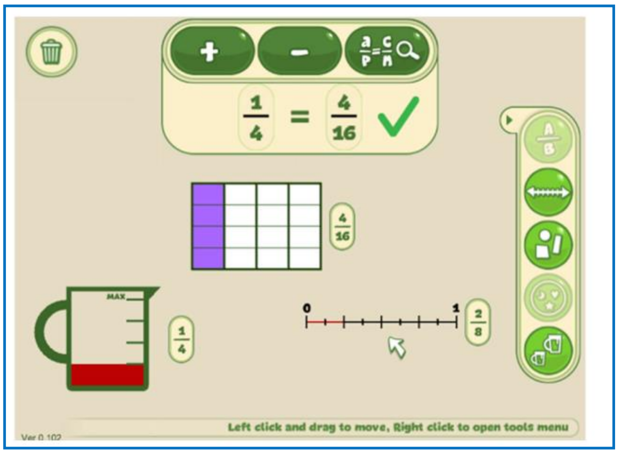

Fig. 6- Fractions Lab showing three representations for 1/4

Regarding the understanding of dividing a unit into equal parts, in their research Olive and Vomvoridi (2006) suggest teachers to avoid incorrect representations, where the division of the fractional unit in equal parts (see Fig. 7) is not frequently observed, which may lead students to believe that diving the unit in equal parts is not necessary. In addition, Tobias (2013) claims that the understanding of the equidivision of a unit into parts depend on the language which the teachers use for defining the whole (e.g. the distinction among of a, of one, of each, and of the).

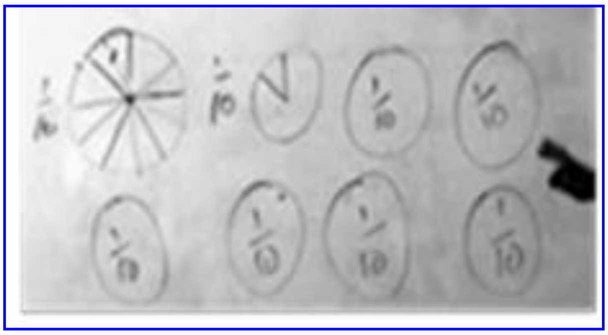

Fig. 7 -Representation about to how to distribute 8 pizzas in 10 people

As far as improper fractions are concerned, Hackenberg (2007) used the JavaBars software on the approximation of the notion, noting the importance of the ability to create improper fractions for placing numbers on the number line, for the construction of fractional numbers that lead the way to developing a sense of consistency and continuity to the numbers (see Fig. 8).

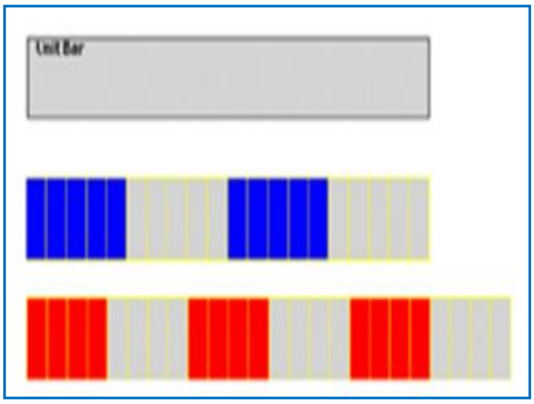

Fig. 8 - Representation of 6/5 on JavaBars

Methods

The researches followed a qualitative, quantitative approach. Moreover, a content analysis and case study were carried out. Thus, a triangulation was formed, which was methodological, temporal, topical and theoretical in order to achieve stabilization of findings (Cohen et al. 2011).

Aim of research

The main purpose of the present study is to provide instructional practices over time implemented by the research team and its positive effect on fraction understanding and development of students' self-confidence in mathematics.

Participants

The sample of the study involved 54 students of the 5th and 6th grade of elementary school in Greece, age 11-12 years. The sample selection was stratified and symptomatic according to the purposes and needs of the research.

Instrument

Regarding the data collection methods, questionnaires and tests written by researchers themselves were used, which reached their final form after pilot studies. Furthermore, instructions, some semi-structured interviews, video recordings, observation and literature study were employed. To achieve the aims of research, students answered a questionnaire – before and after of teachings – which were prepared by our research team to order to be examined the perceptions and the difficulties of pupils at the concept of fractions. The essay contained two parts. The first part of the questionnaire included 22 beliefs questions and second part included seven exercises about fractions.

Data analysis

To analyze the survey data, and in addition to the descriptive analysis, the Statistical Implicative Analysis by Gras, using the CHIC (Cohesive Hierarchical Implicative Classification) software (Gras et al., 1997) and Microsoft Excel program were used. The implication analysis of data was performed through similarity diagrams, in which the variables were associated with each other depending on the similarity or non-similarity they present. Variables in whose solution the subjects behave similarly are grouped together. In addition, the implication analysis of data was performed through implicative graph which presents the variables were associated with each other with implications which are valid at level of significance of 99%.

Variables of research

The variables were defined as a combination of letters and one number. The letters indicate the initial of concept which is examined. For example, the variable NLi5a is composed of the initial proposal “Number Line” because the locating a number on a number line is examined and number 5a indicates the question of questionnaire. According to the implicative analysis, equivalent to a value of 1 was assigned to every item if the answer is correct and 0 if the answer is wrong or missing.

Instructive interventions

The content and activities of teachings were determined by the findings of timeless researches conducted since 2011 to this day and help students reduce the difficulties they face with the concept of fraction as a representation on the geometric model of number line, as well as with the concepts of unit division in equal parts and improper fractions with the help of multiple representations with the ultimate goal of developing students’ self-confidence in mathematics.

Participants were 54 fifth and sixth-grade students (11–12 years old). Three classes were designated-1 class of fifth and 2 classes of sixth grade students. The sample selection of fifth glass and one of two classes of sixth-grade students were symptomatic. The sample selection of the second sixth class was stratified. In particular, in this class participated nine students: three students with high performance at school mathematics, tree students with mediocre performance at school mathematics and three students with low performance at school mathematics.

The teaching interventions were made at different times for each class and they last for two weeks each one. The instructive interventions were implementing by our research team and were divided in 7 phases.

The first phase

The first phase involved completing written essays before the lectures. The essay contained two parts. The first part of the questionnaire included 22 beliefs questions and second part included seven exercises about fractions and specifically about the notions of equal parts of the fractional unit, improper fractions and sequences of rational numbers on the geometric model of number line. The aim of this phase is to be examined the students' fraction knowledge and their beliefs about mathematics before the start of the activities. In this phase participated all three classes of the sample.

The second phase

The second phase involved semi-structured interviews of the participating students before the start of the activities. In this phase participated only the class of sixth grade whose sample was stratified (9 students).

The third phase

The third phase involved lectures that had as a target to expose students to as many multiple representations as possible (see Fig. 9a) and were about the concepts of equal parts and improper fractions. This phase aims to students' conceptual development about unit fraction and about the equidivision of a unit into parts. In addition, this phase aims to develop the students' ability to translate from one representation of the concept of fraction to another.

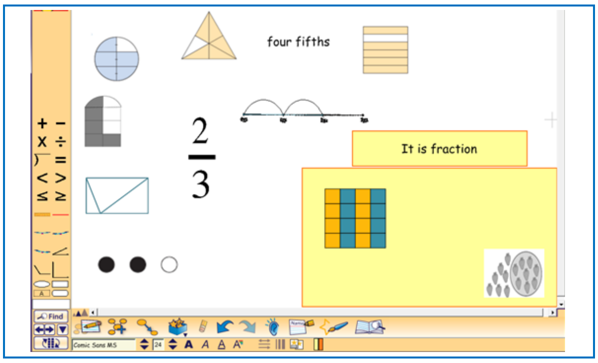

Fig. 9a - Multiple representations during 3th phase of lectures

The presentation of the representations was made with a variety of software (e.g., Microsoft PowerPoint, Microsoft Word software) and included 55 multiple representations in total. In these multiple representations included pictures of fraction unit which were not division into equal part (see Fig. 9b). In this phase participated all three classes of the sample.

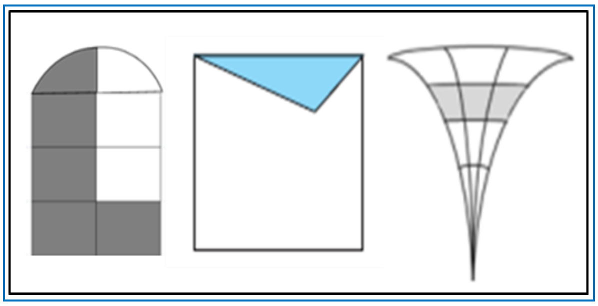

Fig. 9b – Representation of unit which were not division into equal part

The fourth phase

The fourth phase included lectures that were about putting fractions on the number line with the use of experiential representation and representation in a computing environment with the use of the software ConceptuaMath → Place Fractions on the Number Line (see Fig. 10b). This software it is possible to represent a fraction with variety of representations (e.g., symbol, number line, decimals, percent etc.). This phase aims to take students should be capable of locating a number on a number line and, conversely, be able to identify a number represented by a certain point on the number line. In addition, it aims to develop students' knowledge that between any two fractions there is an infinite number of fractions. In this phase participated all three classes of the sample.

This phase begins with a rope was placed on the board representing the number line on which the points 0, 1 and 2 were set (see Fig. 10a). Several tabs were given to students showing different fractions, either improper, proper, whole etc. and each student came to the board in order to place his/her tab correctly on the number line, with reference to existing points, at the beginning only 0,1 and 2, and then to the students' tabs that had been placed on the number line as well. Before the students placed their tab in the number line, they had to report to their classmates, externalising their thoughts, the reason why they intended to place the tab at this point of the number line. Some externalizations of ideas of the students' arguments are fowling:

For 3/4: The numerator is smaller than the denominator, so my fraction is smaller than the unit, so I will place it before 1.

For 299/299: The numerator is equal to the denominator, so it is equal to the unit, so I'll place it under 1.

For 2/150: 2 is very far from 150 which is my unit, so my fraction is close to 0.

For 1/4 (students had already placed 1/5 and 1/3 on the number line): My fraction has the same numerator with 1/5 and 1/3 so it goes between them, since in fractions with same numerator, the one with the smallest denominator is the greatest.

For 149/150: My fraction is very close to the unit, so I'll put it next to 1.

For 8/9: My fraction is very close to unit, it only needs one piece to be 9/9 but this piece is larger than the piece of 149/150 which also needs one piece to be 150/150 so I'll put it next to 1, before 149/150.

For 5/4: In my fraction the numerator is greater than the denominator, so my fraction is greater than the unit so I'll put it after 1.

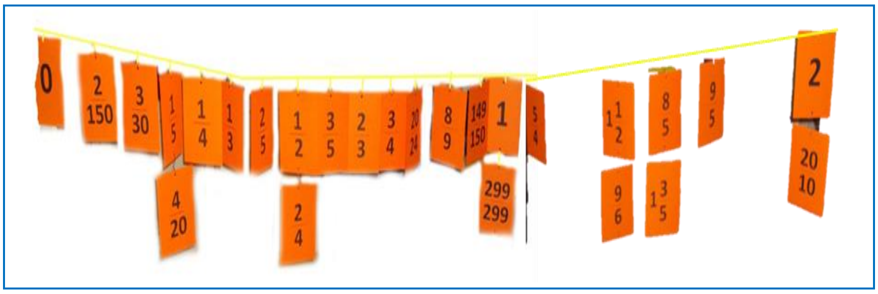

Fig. 10a – The finale form of activities for number line using experiential representation

In any explanation given by the students for their choice, a debate takes place first between the group and then between all groups for whether the choice of each student is correct or not. The confirmation, questions and errors are corrected by the students themselves, using the software Conceptual Math → Place Fractions on a Number Line (see Fig. 10b).

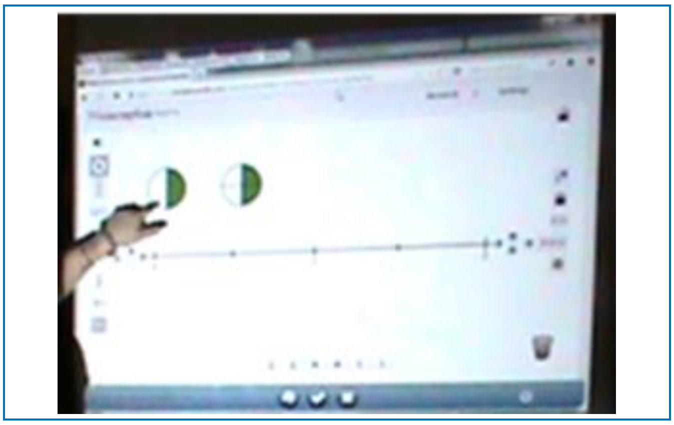

Fig. 10b – Activities for number line using the software ConceptualMath

The fifth phase

The fifth phase included the application of Fraction Battles Software which was created by the research team (see Fig. 11). This software was about the concepts of the equal parts of a unit, of improper fractions and of the classification of rational numbers on the geometric model of the number line. The software's target was to familiarize students with rational numbers and help them reduce difficulties they face with fractions with the assistance of multiple representations on which the added value of the software through a variety of activities of a dynamic multimedia environment. In this phase participated all three classes of the sample.

For its design, all findings from longitudinal surveys from 2011 to 2015 were considered. These researches were carried out by our research team and they investigated the difficulties faced by students in primary and secondary education over rational numbers and the perceptions of prospective teachers over rational numbers, the structure and content of textbooks of mathematics in primary school, as well as on the teaching suggestions and approaches that several researchers have proposed on an international level. This means that every activity of the game is intended to cover a specific difficulty that students face with fractions.

Fig. 11 – Home page of Fraction Battles software

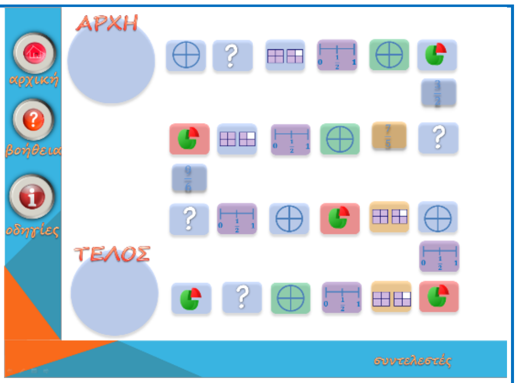

Fig. 12 – Digital dashboard of Fraction Battles

In order to win in Fraction Battles, students must arrive at the finish, throwing the dice and following the route shown in figure (Fig. 12). The route includes 27 points/activities. Each time a student stops at some of these points, he is asked to answer the question/activity by clicking on the corresponding position. If he answers correctly, he continues, otherwise he waits for his turn again. Each activity is designed to refute some of the difficulties students have in rational numbers, as highlighted by previous research. In addition, activities are graded by difficulty (see Fig. 13).

Fig. 13a – Indicative activities of Fraction Battles Software: activity for translating from one representation of the concept of fraction to another

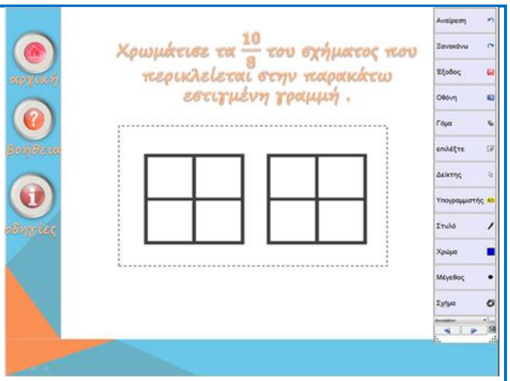

Fig. 13b – Indicative activities of Fraction Battles Software: activity for improper fractions

The sixth phase

The 6th phase involved completing the written essays at the end of the learning process. The essay contained two parts, like the first phase. The aim of this phase is to be examined the students' fraction knowledge and their beliefs about mathematics after at the end of the learning process. In this phase participated all three classes of the sample.

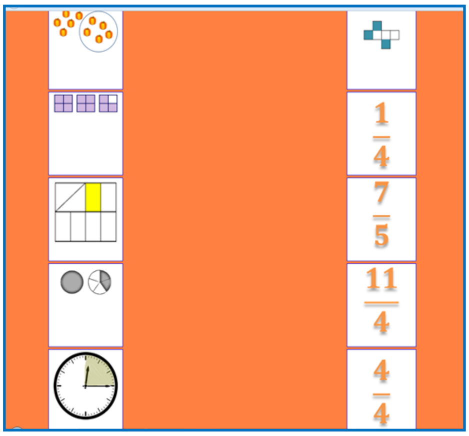

Fig. 13c – Indicative activities of Fraction Battles Software: activity for the equidivision of a unit into parts

The sixth phase

The 6th phase involved completing the written essays at the end of the learning process. The essay contained two parts, like the first phase. The aim of this phase is to be examined the students' fraction knowledge and their beliefs about mathematics after at the end of the learning process. In this phase participated all three classes of the sample.

The seventh phase

The 7th phase involved semi-structured interviews of the participating students (students of second phase) after the instructional interventions.

Results and findings

The implication analysis of data was performed through similarity tree and implicative graph. In similarity tree (see Fig. 14, 15) the variables were associated with each other depending on the similarity or non-similarity they present. Variables in whose solution the subjects behave similarly are grouped together. In implicative graph (see Fig. 16) the variables were associated with each other with implications which are valid at level of significance of 99%. The implication Task 1 → Task 2 means that the success in Task 1 involves success in Task 2 and the failure to Task 2 entails failure in Task 1.

After the teaching interventions, an improvement in students' performance was observed from data analysis (see Fig. 14, 15). In particular, in finding fractions on the number line, the success rate was initially 25% (NLi5a, NLi5b, NLi6ei) and while in the post-activities rates, it rose to 84%. On the notion of unit division into equal parts, we also observed increased success rates, since in exercises concerning recognizing fraction in shapes that were not divided into equal parts (UnFD1a, UnFD1b, UnFD2a, UnFD2d, UnFD2st), rates from 14% in pre-activities, rose to 89% in post-activities. Noteworthy is that the students stated during interviews before the teachings that it is not necessarily fractional unit is divided into equal parts.

In additional, we observed increased success rates in the notion of improper fractions as well, as they rose from 29% to 75% (ImpP7, ImpD2z, ImpD1d) in recognizing improper fraction from a diagram. During interviews before the teachings the many students stated that they did not know what improper fraction means.

Similarity Tree

How this positive change in student performance in rational numbers due to multiple representations is possible to boost their self-confidence in mathematics?

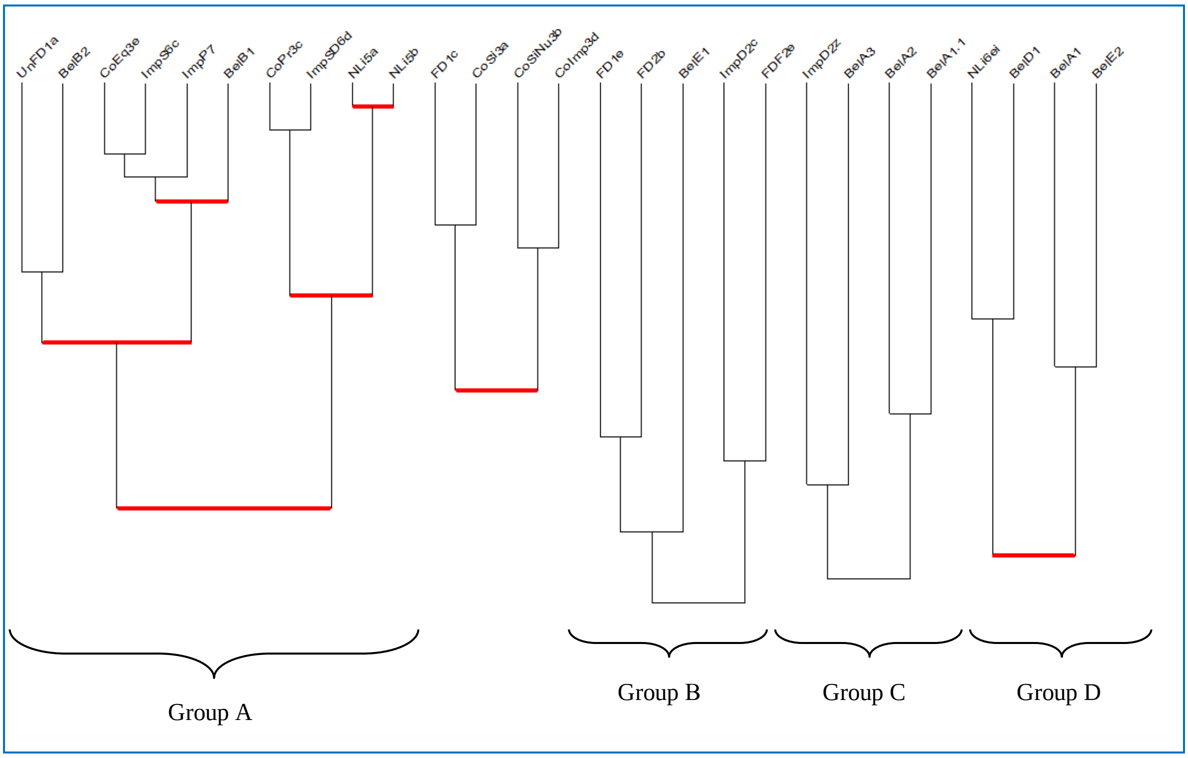

Fig. 14 - Similarity tree 1-Results of pre-test

According to similarity tree 1 (see Fig. 14), which presents the task before the instructive interventions, it is possible to distinguish observe four groups. The first group, group A consists mainly of the variables that present the tasks which are related to the concepts of the unit's division in equal parts and the concept of the improper fractions ((UnFD1a BelB2) (((CoEq3e ImpS6c) ImpP7) BelB1)). These variables are related to the variables BelB2 and BelB1, which concern the belief of students that mathematics would be more attractive to them if they were not so difficult.

From the third group (group C, Fig. 14) it seems that the understanding of improper fraction relates to development of students' self-confidence in mathematics ((ImpD2z BelA3) (BelA2 BelA1.1)). In particular, the students, who stated that they like mathematics and that they are important in human life, achieve to a high success rate in tasks about improper fractions.

On the other hand, according to group D (see Fig. 14) we can observe that the ability of classification of fractions as a representation on the number line, associated with the variables

BelA1, BelD1, BelE2 which concern three categories of statements. The first statement is that students like mathematics (BelA1). The second statement concerns the teaching method (BelD1) and third statement concerns the satisfaction of students according to their performance in mathematics (BelE2). In other words, the students who stated that they like to learn mathematics and that their teachers used examples from everyday life to solve problems and they were satisfied from their performance in mathematics, these students had better performances in tasks about classification of fractions as a representation on the number line.

Comparing similarity tree 1-results of pre-test (see Fig. 14) and Similarity tree 2-results of post-test (Fig. 15), we observe that the improvement of student performance after the explicit instructive interventions using multiple representations positively reinforced the attitude of students towards Mathematics.

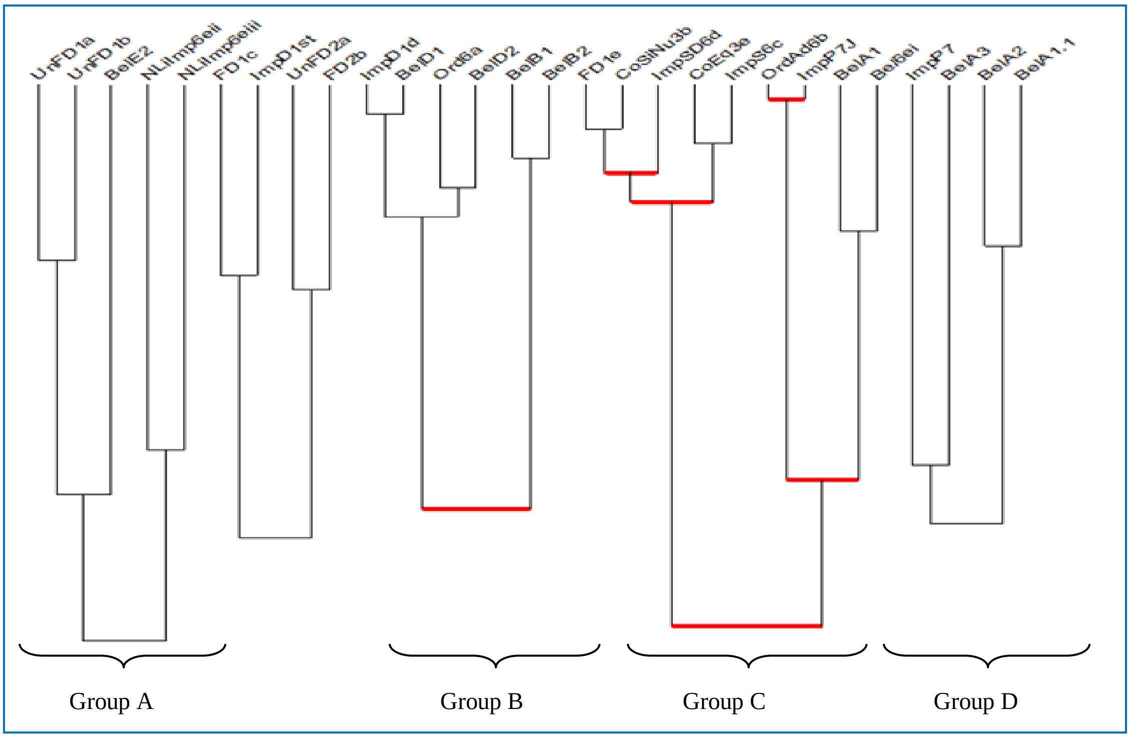

Fig. 15 - Similarity tree 2- Results of post-test

In particular, in group A of similarity tree 2 (see Fig. 15) we observe that the improvement on student's performance on the tasks which the fractional unit presented with diagrammatic representation and it's not divided in equal parts encourages mostly students who stated that they are satisfied from their performance in mathematics ((UnFD1a UnFD1b) BelE2). That change of attitude led to better student's performance in placing improper fractions on the number line (((UnFD1a UnFD1b) BelE2) (NLiImp6eii NLiImp6eiii)).

On group B of similarity tree 2 (see Fig. 15) we observe that the change on the way of teaching with the use of multiple representations improves student's performances on the concept of the improper fraction and on finding a fraction between two fractions ((ImpD1d BelD1) (Ord6a BelD2)). This improvement on student's performance led to the change of their belief that mathematics are not so difficult (((ImpD1d BelD1) (Ord6a BelD2)) (BelB1 BelB2)).

In addition, the improvement on student's performance on more complicated tasks (group C, Fig. 15) such as explaining their solutions of problems on improper fractions, changes the student's attitude towards mathematics and they stated that they like to learn mathematics and that it's important for them to be good at mathematics ((OrdAd6b ImpP7J) (BelA1 BelE1)).

Same as above, group D, Figure 15, presents that the improvement on solving problems of improper fractions boost the state of students that they like mathematics and that mathematics are important on our lives ((ImpP7 BelA3) (BelA2 BelA1.1)).

Implicative graph

The above findings are reinforced by the implicative graph (see Fig. 16) which presents the variables were associated with each other with implications which are valid at level of significance of 99%. More specifically, a positive attitude in mathematics (BelB1) helps students to solve tasks involving both the classification of fractions (CoEqe) and solve problems with improper fractions. Also, the ability of students to solve the above tasks implicates positive change in students' beliefs about mathematics at three levels, at the level of attitudes (BelA1.1, BelA2), at the level of beliefs (BelB2) and at the level of encouragement (BelE2). In other words, students who have improved their performance on these tasks stated that they like mathematics, that mathematics is not so difficult as originally stated and they stated more satisfied with their performance in mathematics. The important finding is that the students after the learning process develop a positive attitude in mathematics (BelB1) which helps them to solve difficult tasks.

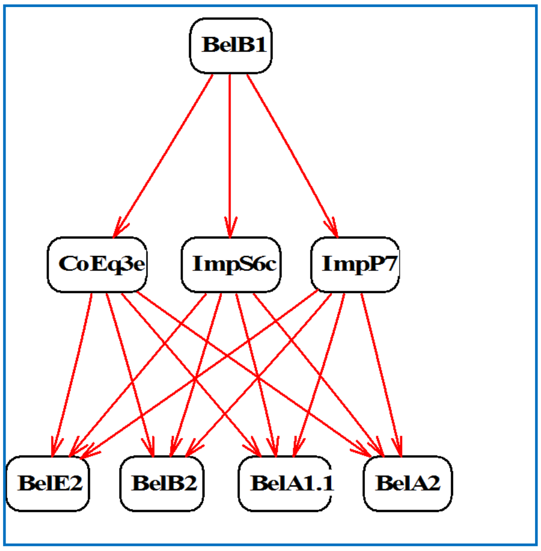

Fig. 16 - Implicative graph of post-test

Conclusion

Rational numbers are an important part of our students' mathematical literacy, as their understanding further contributes to Knowledge other mathematical notions (Empson et al., 2011; Jordan et al., 2013; Hackenberg, 2013; Avgerinos, Vlachou, 2013; Lee, Hackenberg, 2014; Lee, Shin, 2015; Boyce, Norton, 2016). For this reason, many researchers move in this area by investigating the students' difficulties over rational number. Several of these studies share a common component: the idea that the way of teaching is a key factor influencing the future development of the notion of understanding in students' perceptions (Streefland, 1991; Sfard, 1991; Lo, 1993; Chen, Li, 2009; Rønning, 2013; Howe et al., 2015; Dreher et al., 2016; Petakos, 2016) and that the use of multiple representations can play productive role in teaching and learning of mathematics (Janvier, 1987; Dreher, Kuntze, 2015; Shahbari, Peled, 2015; Jacobson, Izsak, 2015; Deliyianni et al., 2016; Hansen et al., 2016).

Hence, despite the research efforts, the constant changes in books, the suggestions etc. students still face the same difficulties and the student population present the same discouraging picture concerning rational numbers in all levels of education. The present study provided instructional practices how the use of multiple representations helps students to reduce their difficulties on fractions and its positive effect on fraction understanding and development of students' self-confidence in mathematics.

The analysis of the collected data both similarity statistical method and implicative graph showed that multiple representations helped enough students to reduce their difficulties about classification of fractions as a representation on the number line, as well as the concepts of the unit's division in equal parts and the concept of the improper fractions.

The reduction of difficulties of students seems to amplify students' self-esteem about mathematics, as they stated to be more satisfied by their performance and expressed more interest for the subject. In particular, the students stated, on a 75% percentage of the respondents, that they are more satisfied by their performance on mathematics after the lectures with the use of multiple representations. In additional, students that stated that they don't like learning mathematics (those students showed low performance on mathematics at school), after the interventions there was a change on their convictions, as they stated that they like mathematics.

The more important finding is that the students after the learning process develop a positive attitude in mathematics which helps them to solve difficult tasks such as tasks with improper fractions. This change of positive attitude in mathematics was affected – according to statements of students in interviews – by better understanding. So, the findings of present research may help researchers and teachers to understand the significance, for some students, of use multiple representations and may help researchers to extend the use of representations to other mathematical concepts.

Author Contributions

All authors designed the study. In particular, E. Aygerinos, and R. Vlachou created the application of Fraction Battles Software and activities of the teaching interventions. In additional, E. Aygerinos, and R. Vlachou collected the data, they analyzed the data and they wrote the manuscript. R. Vlachou made the teaching interventions.

Declaration of Conflicting Interests

The authors declared that they had no conflicts of interest with respect to their authorship or the publication of this article.

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