article

ISSN 2421-7247

Basic geometric-trigonometric equalities and problem-solving: Linear and angular elements

Panagiote Ligouras

Abstract. *This paper presents some equalities that we consider useful for the resolution of general high school level geometry problems. The presented equalities involve the use of linear elements and angular elements at the same time. The objective of this work is to monitor, on a small scale, the skills and knowledge acquired by students in the eleventh and twelfth school year (L11 & L12) in an Italian region named Apulia. This goal has been achieved by mean of different exercises where we asked 27 students:*

- *recognise the structure of proposed equality;*
- *resolve expressions with equalities.*

Results are reported and discussed.

Key words. *Mathematics, Teaching, Geometric equalities, Trigonometric equalities, Angular elements, Linear elements, Problem solving.*

Sommario. *In questo lavoro si presentano alcune uguaglianze che riteniamo utili per la risoluzione di problemi di carattere geometrico a livello di scuola secondaria superiore di secondo grado. Le uguaglianze presentate coinvolgono elementi lineari ed elementi angolari contemporaneamente. L'obiettivo del lavoro è quello di monitorare, in scala molto ridotta, il grado di competenze acquisite dagli studenti dell'undicesimo e del dodicesimo anno di scolarità (L11 & L12) nella regione italiana denominata Puglia. Questo obbiettivo è stato raggiunto tramite diversi esercizi nei quali è stato chiesto a 27 studenti di:*

- *riconoscere la struttura di uguaglianze proposte;*
- *risolvere espressioni con uguaglianze.*

In conclusione, i risultati sono stati presentati e discussi.

Parole chiave. *Matematica, Didattica, Uguaglianze trigonometriche, Uguaglianze geometriche, Elementi angolari, Elementi lineari, Problem-solving.*

Introduction

This article is part of two phase research previously presented (Ligouras, 2017a; Ligouras, 2017b). The first and second phase of the research considered the learning of groups of students who attended a course on geometric equalities concerning only linear elements or only angular elements. The current paper initially presents a collection of equalities that, if used correctly, can facilitate the resolution of many problems of geometry and trigonometry. In this third phase, geometrical and trigonometric equalities, involving linear and angular elements at the same time, were investigated.

In general, the learning process of angular elements' concepts represents a higher difficulty for students with the respect of the ones that uses only linear elements.

Concepts and problems that simultaneously involve linear elements and angular elements represent a greater difficulty for students than learning and consciously applying concepts involving only linear elements or only angular elements.

The equalities have been experimented with groups of students of the third (L11) and of the fourth year (L12) of some high schools of the Apulia region, during their preparation for the mathematical competitions, in the last five years.

The students of the course attended different studies.

The course lasted 8 hours, distributed over two consecutive weeks.

We presented to the students only equations with linear and angular elements. The purpose of the research that we describe in this paper is to explore the skills acquired and the degree of self-regulation achieved by the pupils during their studies in Italian schools. We believe that these two factors stand out better on problematic issues not studied at school.

The primary purpose of the research was to understand if the Italian school:

- prepares students for the acquisition of skills related to problem-solving;
- offers an adequate degree of acquisition of skills related to problem-solving;
- offers the opportunity for students to build meaningful learning in mathematics.

Furthermore, the students' aptitude to work in small groups of three monitored in tasks requiring knowledge not studied at school.

The difference in performance between male and female pupils is also observed.

The students involved in the experimentation described below are different from those involved in the first and second article already presented (Ligouras, 2017a & 2017b).

All the students who participated in the experimentation have a passion for mathematics, high performances and the right motivation to improve them.

Description

Below is a summary of the activities performed and evaluated for this experimentation. To allow a fair comparison of the previous experiment outcomes (Ligouras, 2017a & 2017b) and the ones described in this paper, the experiment content is the same amongst the different publications.

Notations

The figure (Fig.1) shows the annotations useful for reading the basic geometric equations that follow. The annotations provided to each student who participated in the experience.

notations

A, B, C	vertices of a triangle *ABC*	***s***	Semi-perimeter of a triangle
a, b, c	sides *BC, CA, AB*	**[*ABC*]**	Area of triangle *ABC*
α, β, γ	angles of a triangle *ABC*	h_a, h_b, h_c	Altitudes of a triangle *ABC*
R	Radius of circumcircle	m_a, m_b, m_c	Medians of a triangle *ABC*
r	Radius of incircle	w_a, w_b, w_c	Angle-bisectors of a triangle *ABC*
r_a, r_b, r_c	Radii of excircles of a triangle *ABC*	***O***	Circumcentre of a triangle *ABC*
I	Incentre of a triangle *ABC*	***H***	Orthocentre of a triangle *ABC*
I_a, I_b, I_c	Excentres of a triangle *ABC*	***G***	Centroid of a triangle *ABC*

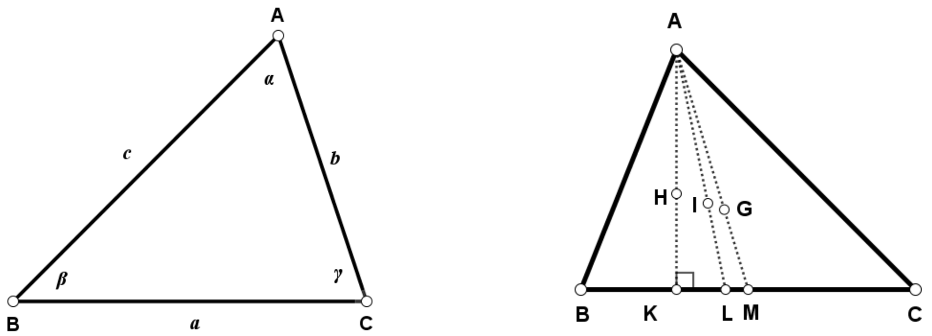

Fig. 1 – Notations

Linear and Angular elements

To perform the experimentation, we used the following 98 geometric equations as a material. This collection has been created over the past five years by examining the formulas used to create the problems of national and international mathematical competitions. For their collection it was very useful Internet but also some documents, articles and books (Andreescu & Feng, 2004, Anonymous, 1904, Altshiller-Court, 2007; Bottema et All., 1969; Engel, 1998; Hang & Wang, 2017; Hobson, 2005; Larson, 1983; Ligouras, 2008; Prasolov & Tikhomirov, 2001; Shariguin, 1989; Zeitz, 2006; Todhunter, 2016).

basic linear and angular elements		
c.1	$a = 2R \cdot \sin\alpha$	
c.2	$\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma} = 2R$	(Law of sines)
c.3	$a = b \cdot \cos\gamma + c \cdot \cos\beta$	
c.4	$a = \dfrac{b \cdot \cos\beta + c \cdot \cos\gamma}{\cos(\beta - \gamma)}$	

c.5	$a=\frac{b\cdot\sin\beta-c\cdot\sin\gamma}{\sin(\beta-\gamma)}$	
c.6	$a^2=b^2+c^2-2bc\cdot\cos\alpha$	(Law of cosines)
c.7	$\sin\alpha=\frac{2}{bc}\cdot\sqrt{s(s-a)(s-b)(s-c)}$	
c.8	$\sin\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$	
c.9	$\cos\frac{\alpha}{2}=\sqrt{\frac{s(s-a)}{bc}}$	
c.10	$\tan\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{r}{s-a}$	
c.11	$\cot\frac{\alpha}{2}=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}=\frac{s-a}{r}$	
c.12	$a=s\cdot\frac{\sin\frac{\alpha}{2}}{\cos\frac{\beta}{2}\cdot\cos\frac{\gamma}{2}}$	
c.13	$\frac{b+c}{a}=\frac{\sin\beta+\sin\gamma}{\sin\alpha}$	
c.14	$\frac{b+c}{a}=\frac{2\cos\frac{\beta}{2}\cdot\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}}-1$	
c.15	$\frac{b+c}{a}=\frac{\cos\frac{\beta-\gamma}{2}}{\sin\frac{\alpha}{2}}$	
c.16	$\frac{b-c}{a}=\frac{\sin\frac{\beta-\gamma}{2}}{\cos\frac{\alpha}{2}}$	
c.17	$\frac{b-c}{a}=\frac{\cos\gamma-\cos\beta}{2\cos^2\frac{\alpha}{2}}$	
c.18	$\frac{a-b}{a+b}=\frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}}=\tan\frac{\alpha-\beta}{2}\cdot\tan\frac{\gamma}{2}$	(Neper)
c.19	$\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}=\frac{a^2-b^2}{c^2}$	

c.20 $$\frac{\tan\alpha}{\tan\beta}=\frac{c^2+a^2-b^2}{b^2+c^2-a^2}$$

c.21 $$a\cdot\sin(\beta-\gamma)+b\cdot\sin(\gamma-\alpha)+c\cdot\sin(\alpha-\beta)=0$$

c.22 $$\frac{a^2\cdot\sin(\beta-\gamma)}{\sin\alpha}+\frac{b^2\cdot\sin(\gamma-\alpha)}{\sin\beta}+\frac{c^2\cdot\sin(\alpha-\beta)}{\sin\gamma}=0$$

c.23 $$\frac{a\cdot\sin\left(\frac{\beta-\gamma}{2}\right)}{\sin\frac{\alpha}{2}}+\frac{b\cdot\sin\left(\frac{\gamma-\alpha}{2}\right)}{\sin\frac{\beta}{2}}+\frac{c\cdot\sin\left(\frac{\alpha-\beta}{2}\right)}{\sin\frac{\gamma}{2}}=0$$

c.24 $$\frac{a\cdot\sin\left(\frac{\beta-\gamma}{2}\right)}{\cos\frac{\alpha}{2}}+\frac{b\cdot\sin\left(\frac{\gamma-\alpha}{2}\right)}{\cos\frac{\beta}{2}}+\frac{c\cdot\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\frac{\gamma}{2}}=0$$

c.25 $$\frac{a^2\cdot\sin(\beta-\gamma)}{\sin\beta+\sin\gamma}+\frac{b^2\cdot\sin(\gamma-\alpha)}{\sin\gamma+\sin\alpha}+\frac{c^2\cdot\sin(\alpha-\beta)}{\sin\alpha+\sin\beta}=0$$

c.26 $$\frac{1}{a}\cdot\cos^2\frac{\alpha}{2}+\frac{1}{b}\cdot\cos^2\frac{\beta}{2}+\frac{1}{c}\cdot\cos^2\frac{\gamma}{2}=\frac{s^2}{abc}$$

c.27 $$(a-b)\cdot\tan\frac{\alpha+\beta}{2}+(b-c)\cdot\tan\frac{\beta+\gamma}{2}+(c-a)\cdot\tan\frac{\gamma+\alpha}{2}=0$$

c.28 $$(a+b)\cdot\tan\frac{\alpha-\beta}{2}+(b+c)\cdot\tan\frac{\beta-\gamma}{2}+(c+a)\cdot\tan\frac{\gamma-\alpha}{2}=0$$

c.29 $$\frac{\cos\alpha-\cos\beta}{s-c}+\frac{\cos\beta-\cos\gamma}{s-a}+\frac{\cos\gamma-\cos\alpha}{s-b}=0$$

c.30 $$R=\frac{1}{2}\cdot\sqrt[3]{\frac{abc}{\sin\alpha\cdot\sin\beta\cdot\sin\gamma}}$$

c.31 $$\sin\alpha+\sin\beta+\sin\gamma=\frac{2R}{r}\cdot\sin\alpha\cdot\sin\beta\cdot\sin\gamma$$

c.32 $$a\cdot\cos\alpha+b\cdot\cos\beta+c\cdot\cos\gamma=4R\cdot\sin\alpha\cdot\sin\beta\cdot\sin\gamma$$

c.33 $$a\cdot\cos\alpha+b\cdot\cos\beta+c\cdot\cos\gamma=\frac{abc}{2R^2}$$

c.34 $$(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)=\frac{s^2}{2R^2}$$

c.35 $$a^2+b^2+c^2=4R^2\cdot\left(\sin^2\alpha+\sin^2\beta+\sin^2\gamma\right)$$

c.36 $$\frac{a\cdot\sin\alpha+b\cdot\sin\beta+c\cdot\sin\gamma}{a\cdot\cos\alpha+b\cdot\cos\beta+c\cdot\cos\gamma}=R\cdot\frac{a^2+b^2+c^2}{abc}$$

c.37 $$a^2+b^2+c^2=2\left(bc\cdot\cos\alpha+ca\cdot\cos\beta+ab\cdot\cos\gamma\right)$$

c.38 $\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}\cdot\tan\frac{\gamma}{2}=\frac{r}{s}$

c.39 $\cos\frac{\alpha}{2}\cdot\cos\frac{\beta}{2}\cdot\cos\frac{\gamma}{2}=\frac{s}{4R}$

c.40 $\sin\frac{\alpha}{2}\cdot\sin\frac{\beta}{2}\cdot\sin\frac{\gamma}{2}=\frac{r}{4R}$

c.41 $\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=\frac{s}{r}$

c.42 $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\frac{4R+r}{s}$

c.43 $\frac{R}{r}=\frac{a+b+c}{a\cdot\cos\alpha+b\cdot\cos\beta+c\cdot\cos\gamma}$

c.44 $\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cdot\cos\frac{\beta}{2}\cdot\cos\frac{\gamma}{2}=\frac{s}{R}$

c.45 $\sin\alpha\cdot\sin\beta\cdot\sin\gamma=\frac{sr}{2R^2}$

c.46 $\cos\alpha+\cos\beta+\cos\gamma=4\sin\frac{\alpha}{2}\cdot\sin\frac{\beta}{2}\cdot\sin\frac{\gamma}{2}+1=\frac{r}{R}+1$

c.47 $\cos\alpha\cdot\cos\beta\cdot\cos\gamma=\frac{s^2-\left(2R+r\right)^2}{4R^2}$

c.48 $a\cdot\cot\alpha+b\cdot\cot\beta+c\cdot\cot\gamma=2\left(R+r\right)$

c.49 $\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}=1-2\sin\frac{\alpha}{2}\cdot\sin\frac{\beta}{2}\cdot\sin\frac{\gamma}{2}=1-\frac{r}{2R}$

c.50 $h_a=\frac{bc\sin\alpha}{a}=\frac{a\cdot\sin\beta\cdot\sin\gamma}{\sin\alpha}$

c.51 $h_a=2s\cdot\frac{\sin\frac{\beta}{2}\cdot\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$

c.52 $AH=2R\cdot\cos\alpha=a\cdot\cot\alpha$

c.53 $HK=2R\cdot\cos\beta\cdot\cos\gamma$

c.54 $a^2+b^2+c^2=R\cdot\left(h_a\cdot\cos\alpha+h_b\cdot\cos\beta+h_c\cdot\cos\gamma\right)$

c.55 $w_a=2\cdot\frac{bc}{b+c}\cdot\cos\frac{\alpha}{2}$

c.56 $w_{1a}=2\cdot\frac{bc}{b-c}\cdot\sin\frac{\alpha}{2}$

c.57 $AI=\frac{bc}{s}\cdot\cos\frac{\alpha}{2}$

c.58 $IL = \frac{abc}{s(b+c)} \cdot \cos\frac{\alpha}{2}$

c.59 $r_a = s \cdot \tan\frac{\alpha}{2}$

c.60 $r_a = 4R \cdot \sin\frac{\alpha}{2} \cdot \cos\frac{\beta}{2} \cdot \cos\frac{\gamma}{2}$

c.61 $r_a + r = 4R \cdot \sin\frac{\alpha}{2} \cdot \cos\frac{\beta-\gamma}{2}$

c.62 $r_a - r = 4R \cdot \sin^2\frac{\alpha}{2}$

c.63 $r_a + r_b = 4R \cdot \cos^2\frac{\gamma}{2}$

c.64 $r_a - r_b = 4R \cdot \cos\frac{\alpha}{2} \cdot \sin\frac{\alpha-\beta}{2}$

c.65 $r = a \cdot \frac{\sin\frac{\beta}{2} \cdot \sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$

c.66 $r_a = a \cdot \frac{\cos\frac{\beta}{2} \cdot \cos\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$

c.67 $r_a = r \cdot \cot\frac{\beta}{2} \cdot \cot\frac{\gamma}{2}$

c.68 $r = \frac{h_a}{2} \cdot \frac{\sin\frac{\alpha}{2}}{\cos\frac{\beta}{2} \cdot \cos\frac{\gamma}{2}}$

c.69 $r_a = \frac{h_a}{2} \cdot \frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2} \cdot \sin\frac{\gamma}{2}}$

c.70 $\frac{h_a - 2r}{h_a} = \frac{h_a}{h_a + 2r_a} = \tan\frac{\beta}{2} \cdot \tan\frac{\gamma}{2}$

c.71 $AI = \frac{s-a}{\cos\frac{\alpha}{2}} = \frac{r}{\sin\frac{\alpha}{2}}$

c.72 $II_a = \frac{a}{\cos\frac{\alpha}{2}} = 4R \cdot \sin\frac{\alpha}{2}$

c.73 $I_aI_b = \frac{c}{\cos\frac{\gamma}{2}} = 4R\cdot\cos\frac{\gamma}{2}$

c.74 $[ABC] = \frac{bc}{2}\cdot\sin\alpha$

c.75 $[ABC] = \frac{a^2}{2}\cdot\frac{\sin\beta\cdot\sin\gamma}{\sin\alpha}$

c.76 $[ABC] = \frac{abc^2}{2\left(a^2-b^2\right)}\cdot\sin(\alpha-\beta)$

c.77 $[ABC] = \frac{\left(a^2-b^2\right)}{2}\cdot\frac{\sin\alpha\cdot\sin\beta}{\sin(\alpha-\beta)} = \frac{a^2-b^2}{2(\cot\alpha-\cot\beta)}$

c.78 $[ABC] = s^2\cdot\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}\cdot\tan\frac{\gamma}{2}$

c.79 $[ABC] = \frac{abc}{s}\cdot\cos\frac{\alpha}{2}\cdot\cos\frac{\beta}{2}\cdot\cos\frac{\gamma}{2}$

c.80 $[ABC] = \frac{a^2+b^2+c^2}{4(\cot\alpha+\cot\beta+\cot\gamma)}$

c.81 $[ABC] = \frac{a^2\cdot\sin(2\beta)+b^2\cdot\sin(2\alpha)}{4}$

c.82 $[ABC] = R\cdot\sin\beta\cdot\sin\gamma$

c.83 $[ABC] = 2R^2\cdot\sin\alpha\cdot\sin\beta\cdot\sin\gamma$

c.84 $[ABC] = 4Rs\cdot\sin\frac{\alpha}{2}\cdot\sin\frac{\beta}{2}\cdot\sin\frac{\gamma}{2}$

c.85 $[ABC] = \frac{1}{2}R\cdot(a\cdot\cos\alpha+b\cdot\cos\beta+c\cdot\cos\gamma)$

c.86 $[ABC] = Rr\cdot(\sin\alpha+\sin\beta+\sin\gamma)$

c.87 $[ABC] = R\cdot h_a\cdot\sin\alpha$

c.88 $[ABC] = \frac{{m_a}^2+{m_b}^2+{m_c}^2}{3(\cot\alpha+\cot\beta+\cot\gamma)}$

c.89 $[ABC] = \frac{2}{3}\cdot\left({m_b}^2-{m_a}^2\right)\cdot\frac{\sin\alpha\cdot\sin\beta}{\sin(\alpha-\beta)}$

c.90 $[ABC] = \frac{1}{2}\cdot w_a\cdot(b+c)\sin\frac{\alpha}{2} = \frac{1}{2}\cdot w_a\cdot a\cdot\cos\frac{\beta-\gamma}{2}$

c.91 $[ABC] = r^2\cdot\cot\frac{\alpha}{2}\cdot\cot\frac{\beta}{2}\cdot\cot\frac{\gamma}{2}$

c.92	$[ABC] = r_a^2 \cdot \cot\frac{\alpha}{2} \cdot \tan\frac{\beta}{2} \cdot \tan\frac{\gamma}{2}$
c.93	$[ABC] = r^2 \cdot \cot\frac{\alpha}{2} + 2Rr \cdot \sin\alpha$
c.94	$[ABC] = rr_a \cdot \cot\frac{\alpha}{2}$
c.95	$[ABC] = r_b r_c \cdot \tan\frac{\alpha}{2}$
c.96	$[ABC] = \frac{r^2}{2} \cdot \frac{(\sin\alpha + \sin\beta + \sin\gamma)^2}{\sin\alpha \cdot \sin\beta \cdot \sin\gamma}$
c.97	$[ABC] = \frac{b^2 + c^2 - a^2}{4\cot\alpha}$
c.98	$[ABCD] = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \cdot \cos^2\left(\frac{\alpha+\gamma}{2}\right)}$

Fig.2 – Basic geometric-trigonometric equalities with linear and angular elements

Activities and experimentation

As a first activity, the students practised recognising if a specific expression is an immediate consequence of one of those illustrated in the figure (Fig.2). The following figure (Fig.3) presents one of the cards used for the exercise.

Each student had 10 minutes to answer the question.

Proposal n. 4	
la.01	$r_a = 2R \cdot \sin\frac{\alpha}{2} \cdot \cos\frac{\beta}{2} \cdot \cos\frac{\gamma}{2} + \frac{s}{2} \cdot \tan\frac{\alpha}{2}$
la.02	$\tan\frac{\beta}{2} \cdot \tan\frac{\gamma}{2} = \frac{h_a - 2r}{2h_a} + \frac{h_a}{2h_a + 4r_a}$
la.03	$a + b = (a+b) \cdot \cos\gamma + c \cdot (\cos\beta + \cos\alpha)$
<u>la.04</u>	$R \cdot \left(1 - \sin^2\frac{\alpha}{2} - \sin^2\frac{\beta}{2} - \sin^2\frac{\gamma}{2}\right) = \frac{r}{2}$
la.05	$[ABC] = rr_a \cdot \frac{r_b - r_c}{b - c}$
la.06	$\frac{b+c}{a} = \frac{\sin\beta + \sin\gamma}{2\sin\alpha} + \frac{\cos\frac{\beta}{2} \cdot \cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}} - \frac{1}{2}$

la.07	$a^2 \cos\gamma + c^2 \cos\alpha = \frac{a+c}{2b}\left[b^2 + (a-c)^2\right]$
<u>**la.08**</u>	$a = \frac{r \cdot \cos\frac{\alpha}{2}}{\sin\frac{\beta}{2} \cdot \sin\frac{\gamma}{2}}$
la.09	$[ABC] = r^2 \cdot \cot\frac{\alpha}{2} + 2Rr \cdot \sin\alpha$
la.10	$r_a = a \cdot \sec\frac{\alpha}{2} \cdot \cos\frac{\beta}{2} \cdot \cos\frac{\gamma}{2}$
Identify any expressions that are immediate consequences of one of the ninety-eight *basic trigonometric equalities*. Indicate below their ID.	
Answers:	

Fig. 3 – recognition questionnaire

Subsequently, students were offered some cards with problems to verify their ability to recognise the expressions that are constructed by combining two or three geometrical equalities between the 98 of the figure (Fig. 2). The following figure (Fig. 4) presents one of the boards used for the exercise.

Each student had 10 minutes to answer the question.

Proposal n. 6	
k.01	$[ABC] = \frac{b^2 + c^2 - a^2}{4\tan\frac{\beta+\gamma-\alpha}{2}}$
k.02	$\sqrt{(ab)^2 - 4[ABC]^2} + \sqrt{(bc)^2 - 4[ABC]^2} + \sqrt{(ca)^2 - 4[ABC]^2} = \frac{a^2 + b^2 + c^2}{2}$
k.03	$[ABC] = rr_a \cdot \sqrt{\frac{4R - (r_a - r)}{r_a - r}}$
<u>**k.04**</u>	$r_a{}^2 - r_b{}^2 = 16R^2 \cdot \sin\frac{\alpha-\beta}{2} \cdot \cos\frac{\alpha}{2} \cdot \cos^2\frac{\gamma}{2}$
k.05	$w_a = 2R \cdot \frac{\sin\beta \cdot \sin\gamma}{\cos\frac{\beta-\gamma}{2}}$
Identify any expressions that are a consequence of two or three of the ninety-eight expressions provided. Indicate below their ID.	
Answers:	

Fig. 4 – Combination of *basic geometric equalities*

Finally, the students were asked to demonstrate one of the five expressions proposed using their knowledge and the 98 equalities (see Fig. 2). The following figure (Fig.5) presents one of the boards used for group exercise.

Survey n. 2	
m.1	$[ABC]=\frac{(s-a)^2\sin\alpha+(s-b)^2\sin\beta+(s-c)^2\sin\gamma}{2\left(\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}\right)}$
m.2	$\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\cdot\tan\frac{\beta}{2}+\frac{r}{s-b}\cdot\tan\frac{\gamma}{2}+\frac{r_c}{s}\cdot\tan\frac{\alpha}{2}=1$
m.3	$a\cdot(\cos\beta+\cos\gamma)(1+2\cos\alpha)=(b+c)\cdot\left(1+\cos\alpha-2\cos^2\alpha\right)$
m.4	$2s=\frac{a\cdot\cos\alpha+b\cdot\cos\beta+c\cdot\cos\gamma}{\cos\alpha+\cos\beta+\cos\gamma}$
m.5	$\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}+2\cdot\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\frac{r}{CI}\cdot\sin\frac{\beta}{2}=1$
Solve, in groups, one of the following geometric equalities. Basic geometric-trigonometric equalities can use without demonstration.	
Resolution:	

Fig. 5 – Resolution, in groups, geometric equalities

We provided, to perform this exercise, the cards with the equations presented in the two previous works of the author (Ligouras, 2017a, Fig.2 & Ligouras, 2017b, Fig.2). These equalities, if used, must be demonstrated. We made this choice because the information provided in Fig.2 is not sufficient to solve the proposed problems. Furthermore, this research does not intend to measure knowledge but to appreciate the pupils' skills.

Each group had 20 minutes to answer the question.

Finally, the figure (Fig. 6) presents one of the worksheets used, at the end of the path, to monitor the skills acquired (acquired and previous) individually.

Also for the performance of this task the two cards, described above, have been provided to each candidate (Ligouras, 2017a, Fig.2 & Ligouras, 2017b, Fig.2).

Each student has had 35 minutes to answer the question.

Proposal n. 1	
prob.1	$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=\frac{4[ABC]}{R}\cdot\sin\alpha$

prob.2 $$b\cdot\sin(\gamma-\alpha)+c\cdot\sin(\alpha-\beta)=\frac{s\cdot\sin\frac{\alpha}{2}}{\cos\frac{\beta}{2}\cdot\cos\frac{\gamma}{2}}\cdot\sin(\gamma-\beta)$$

prob.3 $$\frac{\cos\beta+\cos\gamma}{\sin\beta+\sin\gamma}\cdot\tan\frac{\beta}{2}\cdot[ABC]=\frac{rr_b r_a}{s}$$

Solve, individually, one of the three previous equalities. Basic geometric-trigonometric equalities can use without demonstration.

Resolution:

Fig. 6 – Individually final test. Solution of geometric equalities

Discussion and results

27 students from L11 and L12 school year participated in this research. Of these, 15 were males and 12 females. During the course topics and insights were discussed that are generally not carried out at school during the period of study in Italy.

The first figure (Fig.7) presents the percentages of the exact answers achieved by the students during the first exercise.

- 77.78% of students have identified that question la.04 is an immediate consequence of c.49 of fundamental geometric equalities. This question had a correct answer from 86.67% of the 15 male students, which corresponds to 13 students and 66.67% of the 12 female students corresponding to 8 people.
- 88.89% identified that question la.08 is a consequence of equality c.65. This question was answered correctly by 93.33% of males, which corresponds to 14 students and 83.33% of female students who correspond to 10 people.

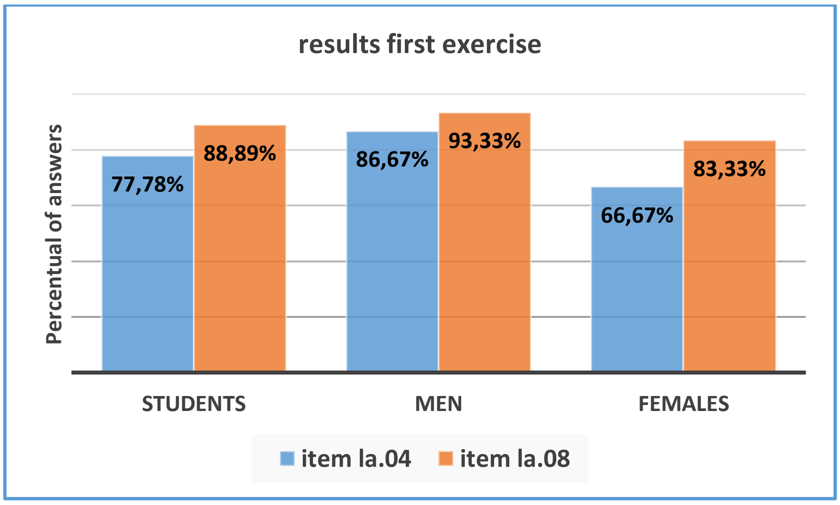

Fig. 7 – Results first exercise

The percentage of responses cannot be considered a good result. The task was only to observe and decide to use only the four fundamental operations of arithmetic. Also, the results of the percentages of correct answers of the other cards were similar and included in a range of ±1.96% compared to the values presented in the graph.

The figure (Fig.8) shows the percentages of the correct and non-correct answers for the second exercise.

The exercise asked to identify the equalities that would result in combinations of two or three basic expressions.

- 7.41% of the students have identified that question k.1 is a consequence of the equality c.97 (see Fig.2). Only two students answer incorrectly.

  To obtain the identity k.1 I need the equality c.97 and the goniometric expression $\cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$. The item is not included in the correct answers.

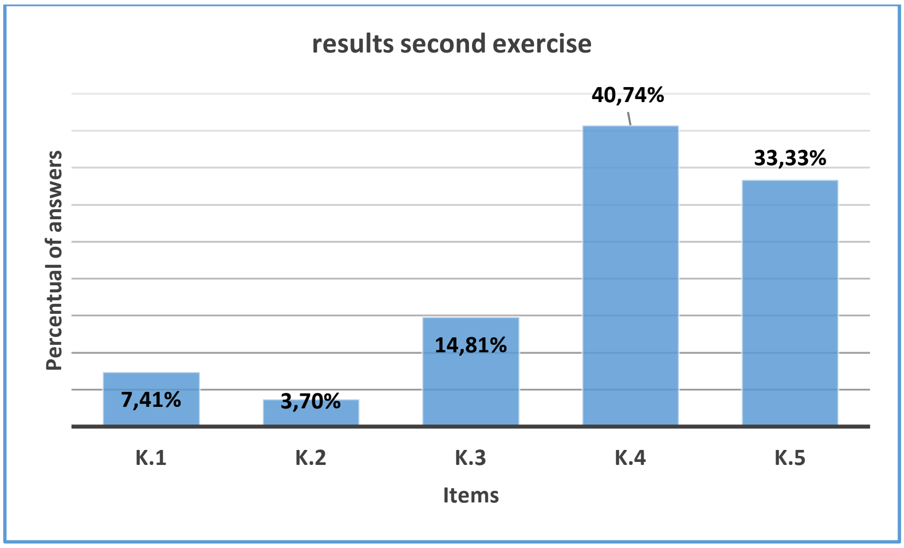

Fig. 8 – Results second exercise

- 3.70% of the students indicated that the question k.2 is a consequence of the question c.80 and c.97. The student answered incorrectly. Item k.2 to be verified needs the following equalities: c.74 and $\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$.
- 14.81% of the students identified that question k.3 is a consequence of questions c.64 and c.94. These four students answered incorrectly. The equality of item k.3 to be verified needs c.64, c.94 and the fundamental expression $\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} = 1$.
- 40.74% of the students indicated that the question k.4 is a consequence of the questions c.64 and c.65. These eleven students answered correctly. The equality can achieve by following simple algebraic operations using items c.64 and c.65.

- 33.33% of the 27 students identified that question k.5 is a consequence of question b.13. These nine students answered correctly. To demonstrate the equality of item k.3 it is sufficient to use the expressions c.82 and c.90.

Only three students out of twenty-seven (which corresponds to 5%) indicated that there are two correct answers. Everyone else has indicated only one solution.

We can think, after reading these results, that even the students of the second phase of experimentation like those of the first phase, cannot read carefully and consequently understand the deliveries of the tasks to be performed. Our conjecture could further deepen (with subsequent research) extending the statistical sample to other Italian students of school age L11 and L12. The exercise asked to identify the equalities that would result in combinations of two or more basic expressions.

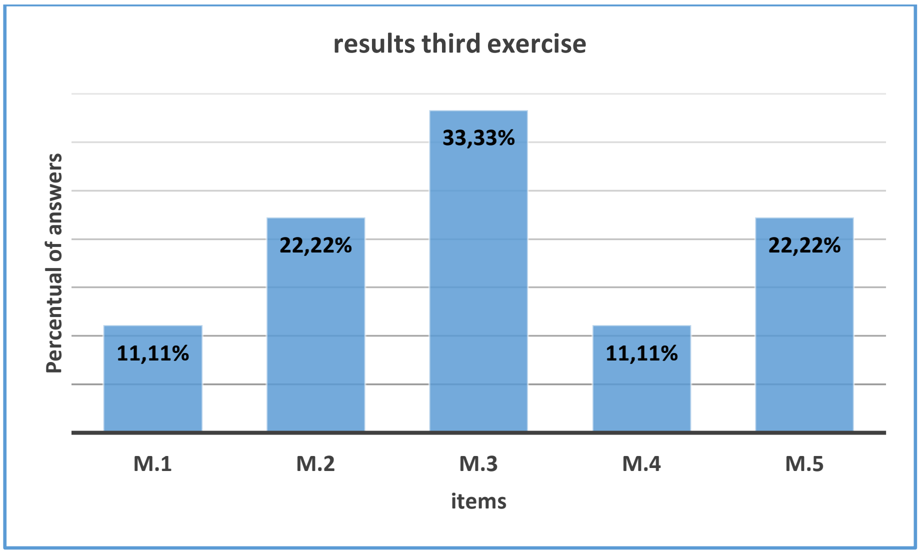

Fig. 9 – Results third exercise

Figure (figure 9) presents the percentages of the nine groups of students who have solved a specific item of the third exercise.

- 11.11% of the groups, which corresponds to 1 group of the nine formats, solved the question m.1. The question presents a medium-high difficulty.
- 22.22% which corresponds to 2 groups, solved the question m.2. The question presents a medium difficulty.
- The 33.33% that corresponds to 3 groups solved the question m.3. The question presents a medium difficulty.
- 11.11% which corresponds to 1 group, solved the question m.4. The question has a medium difficulty.
- 22.22% (corresponds to 2 groups) solved the question m.5. The question presents a medium-high difficulty.

It is important to stress that all the groups have solved one of the five proposed questions successfully.

As can be seen in the last figure (Fig. 10):

- 22.22% of the students, which corresponds to 6 people, solved the question prob.1. The application presents a medium-high level of difficulty. The c.82 is the fundamental equation to be used to solve the proposed problem.
- 48.15% of the students, which corresponds to 13 people, solved the question prob.2. The question does not present a high level of difficulty. The largest number of students chose this question. The c.21 is the fundamental equation to be used to solve the proposed problem.
- The 18.52% that corresponds to 5 students solved the question prob.3. The question presents a higher level of difficulty with than the other two proposed. c.38 and c.95 are the fundamental equations to be used to solve the proposed problem.

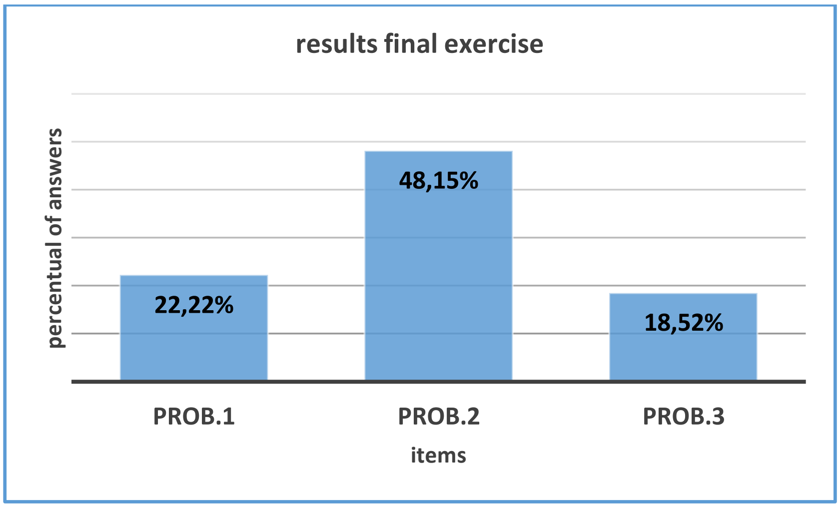

Fig. 10 – Results final exercise

Finally, it should note that the 11.11% that corresponds to 3 students, has not been able to completely or correctly solve any item of those proposed.

Also in the second phase, we did not intervene with the students to understand the motivations of the decision-making process that guided their response choices.

Conclusion

The results presented in this article are in line with the results presented in the papers (Ligouras, 2017a) and (Ligouras, 2017b). The three experiments have shown that the more self-regulated, motivated and more grit (Duckworth, 2016) L11 and L12 students can exploit their knowledge and skills to improve the skills already in their possession and extend them faster, guided by other amongst the most competent peers in the group or the teacher. As demonstrated by researchers, there is a new zone of proximal development outside the student's current area of development. This zone is called (ZPD) *Zone of Proximal Development* (Vygotsky, 2012). Our experiments, as we just said, have shown that, in the groups of more self-regulated pupils, the transition between the *current development zone* and the *proximal development zone* of the student would seem quicker.

The author intends to continue the research following the same basic methodology. This new experiment that will involve a new group of students will have two phases. In the first phase, there will be exercises with the following different types of geometric equality: equalities with linear elements; equality with angular elements and equality with linear and angular elements together. The first phase will create the basis for developing the next phase. The second phase will involve geometric inequalities. In this way - by gradually inserting linear elements, angular elements, linear and angular elements and geometric inequalities - the information field provided is expanded and the requests to the students become more complex. This way of proceeding seems to the authors an excellent path to evaluate the degree of ability of the students in managing an increasing amount of information on increasingly complex problematic situations. Problem solving skills will be tested on topics closer to the world of abstraction then to the factual world but always within the ZPD. Students will be prepared before the final evaluation to have solid bases in the domain of geometric equality and will learn the essential techniques to solve inequalities. Students will not be prepared to solve geometric inequalities before the final evaluation.

None of the students in the group highlighted particular heuristic skills.

Declaration of Conflicting Interests

The author declared that they had no conflicts of interest concerning their authorship or the publication of this article.

References

Andreescu T., Feng Z., (2004). *103 Trigonometry Problems: From the Training of the USA IMO Team*, Birkhauser, Boston.

Anonymous, (1904). *Relations entre les éléments d'un triangle*. Vuibert et Nony, Éditeurs, Paris.

Altshiller-Court N., (2007). *College Geometry: An Introduction to the Modern Geometry of the Triangle and the Circle*, Dover Publications.

Bottema O., Djordjevic R.Z., Janic R.R., Mitrinovic D.S., Vasic P.M., (1969). *Geometric Inequalities*, Wolters-Noordho Publishing Groningen.

Duckworth A., (2016). *Grit. The power of passion and perseverance*, Scribner.

Engel A., (1998). *Problem Solving Strategies*, Springer Verlang.

Hang K.H., Wang H., (2017). *Solving Problems in Geometry: Insights and Strategies*, World Scientific Pub Co Inc.

Hobson E.W., (2005). *A Treatise on Plane and Advanced Trigonometry*, Dover Publications, Inc. Mineola, New York.

Larson L.C., (1983). *Problem-Solving Through Problems*, Springer-Verlag.

Ligouras P., (2017a). *Basic Geometric Equalities and Problem-Solving: Linear Elements*, Experiences of Teaching with Mathematics, Sciences and Technology — ISSN 2421-7247, vol. 3, n. 1, 475-487.

Ligouras P., (2017b). *Basic Geometric Equalities and Problem-Solving: Angular Elements*, Experiences of

Teaching with Mathematics, Sciences and Technology — ISSN 2421-7247, vol. 3, n. 2, 515-526.

Ligouras P., (2010). 1691 *Algebraic Inequalities: Problem Solving – Old and New Problems for the Mathematical Olympiads*, AGA editrice.

Ligouras P., (2008). *Geometrical Olympiad 2008*, AGA editrice, ISBN: 88-95089-11-9 (Italian).

Prasolov V.V., Tikhomirov V.M., (2001). *Geometry*, American Mathematical Society.

Shariguin I., (1989). *Problemas de Geometría Planimetría*. Editorial Mir.

Todhunter I., *Problems and Solutions in Plane Trigonometry (LaTeX Edition): For the use of Colleges and Schools*, Ancient Science Publishers, 2016.

Vygotsky L.S., (2012). *Thought and Language*, 2nd Edition, The MIT Press.

Zeitz P., (2006). *The Art and Craft of Problem Solving*, Wiley International Student edition.

About the Author

Panagiote Ligouras
I.I.S. "Leonardo da Vinci – Galileo Galilei" - Noci (BA)
Currently in service at the Regional School Office (USR-Puglia)
Via Col di Lana, 33, 70011 Alberobello (BA)
ligouras@alice.it
Italy

Teacher of mathematics and computer science. Passionate about mathematical problem-solving, ICT, didactic communication and online and Blended educational activities. It also deals with learning and evaluation processes in various training and system contexts.

Collaborates for years with the Italian Ministry of Education, University and Research (MIUR), with the INDIRE (National Institute of Documentation, Innovation and Educational Research, Italy), with INVALSI (National Institute for the Evaluation of the Educational System of Education and training, Italy) and with the USR Puglia (Regional School Office, Italy).

Trainer accredited in "Evaluation of learning and system" - SNV.

He is the author of numerous scientific papers.

Website: www.takismath.eu

LinkedIn: http://it.linkedin.com/pub/ligouras-panagiote/33/113/a02

Received April 17, 2017; *revised* June 22, 2017; *accepted* October 6, 2017; *published online* December 04, 2018