article

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Basic geometric equalities and problem-solving: Linear elements

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Abstract. *This paper presents some equalities that we consider useful for the resolution of general high school level geometry problems. The presented geometric equalities involve only linear elements. The objective of this work is to monitor, on a very small scale, the skills and knowledge acquired by students in the eleventh year of school in an Italian region named Apulia. This has been done by mean of different exercises where 23 students were asked to recognize and combine the equalities presented. Results are reported and discussed.*

Key words. *Mathematics, Teaching, Geometric equalities, Linear elements, Problem solving.*

Sommario. *In questo lavoro si presentano alcune uguaglianze che riteniamo utili per la risoluzione di problemi di carattere geometrico a livello di scuola secondaria superiore di secondo grado. Le uguaglianze geometriche presentate coinvolgono solo elementi lineari. L'obiettivo del lavoro è quello di monitorare, in scala molto ridotta, il grado di competenze acquisite dagli studenti dell'undicesimo anno di scolarità nella regione italiana denominata Puglia. Questo obbiettivo è stato raggiunto tramite diversi esercizi nei quali è stato chiesto a 23 studenti di riconoscere e combinare le uguaglianze proposte. In conclusione, i risultati sono stati presentati e discussi.*

Parole chiave. *Matematica, Didattica, Uguaglianze geometriche, Elementi lineari, Problem-solving.*

Introduction

This paper presents a rich collection of equalities that can be used to help solve many geometry problems. The equalities have been tested, for the last ten years, on several groups of third and fourth year students coming from different Apulian's high schools. The aim of this study is to understand how well the Italian schools prepare their students for the acquisition of transferable skills like problem-solving and lateral thinking. Furthermore, the students' aptitude to work in small groups of three persons has been monitored.

Description

Below is a summary of the activities carried out and evaluated during this study.

Notations

Figure (Fig. 1) shows the notations used for the following basic geometrical equations. The notations were given to every student involved in the experiment.

notations	
A, B, C	vertices of a triangle
a, b, c	sides *BC*, *CA*, *AB*
α, β, γ	angles of a triangle
h_a, h_b, h_c	altitudes
m_a, m_b, m_c	medians
w_a, w_b, w_c	angle-bisectors
R	radius of circumcircle
r	radius of incircle
s	semi-perimeter
r_a, r_b, r_c	radii of excircles
O	circumcentre
I	incentre
H	orthocentre
G	centroid
I_a, I_b, I_c	excentres
[*ABC*]	area of triangle *ABC*
[*F*]	area of figure *F*
P or ***M***	point in the interior of a triangle

Fig. 1 – Notations

Linear elements

The following ninety basic geometrical equalities were used in order to carry out the experiment. These equations were collected over the last eight years by looking at the formulas used to create the problems of international maths competitions such as IMO, Balkan Mathematical Olympiad, The Nordic Mathematical Contest, Asian Pacific Mathematics Olympiad, Pan-African Mathematical Olympiad, United States of America Mathematical Olympiad. Internet, as well as some papers and books, were also very useful sources (Andreescu & Feng, 2004; Anonymous, 1904; Altshiller-Court, 2007; Bottema et All., 1969; Engel, 1998; Hang & Wang, 2017; Hobson, 2005; Larson, 1983; Ligouras, 2008; Prasolov & Tikhomirov, 2001; Shariguin, 1989; Zeitz, 2006).

basic linear elements	
a.1	$a \cdot h_a = b \cdot h_b = c \cdot h_c = 2 \cdot [ABC]$
a.2	$h_a = \frac{2}{a} \cdot \sqrt{s(s-a)(s-b)(s-c)}$

a.3 $[ABC] = s \cdot r$

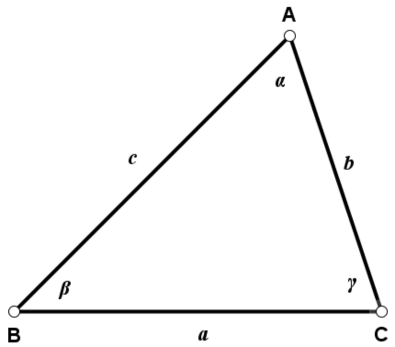
a.4 $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$

a.5 $a \cdot \sin(\beta - \gamma) + b \cdot (\gamma - \alpha)$

a.6 $bc = 2Rh_a$

a.7 $abc = 4R \cdot [ABC]$

a.8 $ab + bc + ca = s^2 + 4Rr + r^2$



a.9 $a^2 + b^2 + c^2 = 2s^2 - 8Rr - 2r^2$

a.10 $h_a + h_b + h_c = \dfrac{ab + bc + ca}{2R}$

a.11 $\dfrac{OI_a}{h_a} + \dfrac{OI_b}{h_b} + \dfrac{OI_c}{h_c} = 1$

a.12 $(h_a + h_b + h_c)\left(\dfrac{1}{h_a} + \dfrac{1}{h_b} + \dfrac{1}{h_c}\right) = (a + b + c)\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)$

a.13 $h_a \cdot AH = \dfrac{b^2 + c^2 - a^2}{2}$

a.14 $h_a \cdot AH + h_b \cdot BH + h_c \cdot CH = \dfrac{a^2 + b^2 + c^2}{2}$

a.15 $AH \cdot HD = BH \cdot HE = CH \cdot HF$

a.16 $AH \cdot BH = 2R \cdot HF$

a.17 $AH \cdot BH + BH \cdot CH + CH \cdot AH = 2R \cdot (HD + HE + HF)$

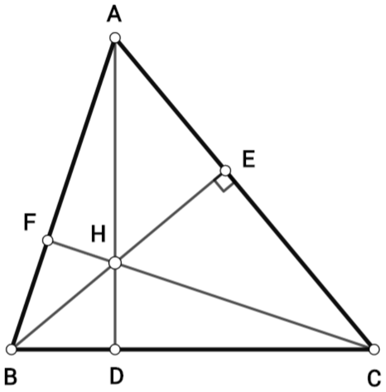
a.18 $AH^2 \cdot BH^2 \cdot CH^2 = 8R^3 \cdot DH \cdot EH \cdot FH$

a.19 $AH = 2 \cdot OX$

a.20 $a^2 + AH^2 = b^2 + BH^2 = c^2 + CH^2 = 4R^2$

a.21 $AH + BH + CH = 2(R + r)$

a.22 $a(OY + OZ) + b(OZ + OX) + c(OX + OY) = 2R \cdot s$



a.23 $4\left(OX\cdot R+OY\cdot OZ\right)=bc$

a.24 $4\left(h_a OY+h_b OZ+h_c OX\right)=a^2+b^2+c^2$

a.25 $4\left(\frac{a}{OX}+\frac{b}{OY}+\frac{c}{OZ}\right)=\dfrac{abc}{OX\cdot OY\cdot OZ}$

a.26 $4\left(OX^2+OY^2+OZ^2\right)=12R^2-\left(a^2+b^2+c^2\right)$

a.27 $OX\left(\dfrac{b}{c}+\dfrac{c}{b}\right)+OY\left(\dfrac{c}{a}+\dfrac{a}{c}\right)+OZ\left(\dfrac{a}{b}+\dfrac{b}{a}\right)=6R$

a.28 $4\left(OX\cdot OY+OY\cdot OZ+OZ\cdot OX\right)=ab+bc+ca-4R\left(R+r\right)$

a.29 $m_a=\dfrac{1}{2}\cdot\sqrt{2\left(b^2+c^2\right)-a^2}$

a.30 $m_a{}^2+m_b{}^2+m_c{}^2=\dfrac{3}{4}\left(a^2+b^2+c^2\right)$

a.31 $16\left(m_a^2m_b^2+m_b^2m_c^2+m_c^2m_a^2\right)=9\left(a^2b^2+b^2c^2+c^2a^2\right)$

a.32 $16\left(m_a^4+m_b^4+m_c^4\right)=9\left(a^4+b^4+c^4\right)$

a.33 $AG=2\cdot GD$

a.34 $w_a=\dfrac{2}{b+c}\cdot\sqrt{s\left(s-a\right)bc}$

a.35 $w_{1a}=\dfrac{2}{b-c}\cdot\sqrt{\left(s-b\right)\left(s-c\right)bc}\qquad\left(with\ b>c\right)$

a.36 $bc=\dfrac{1}{2}\cdot\sqrt{w_a{}^2\left(b+c\right)^2+w_{1a}{}^2\left(b-c\right)^2}$

a.37 $a\cdot AI^2+b\cdot BI^2+c\cdot CI^2=abc$

a.38 $\dfrac{AI\cdot BI\cdot CI}{DI\cdot EI\cdot FI}=\dfrac{\left(a+b\right)\left(b+c\right)\left(c+a\right)}{abc}$

a.39 $r_a=\sqrt{\dfrac{s\left(s-b\right)\left(s-c\right)}{s-a}}$

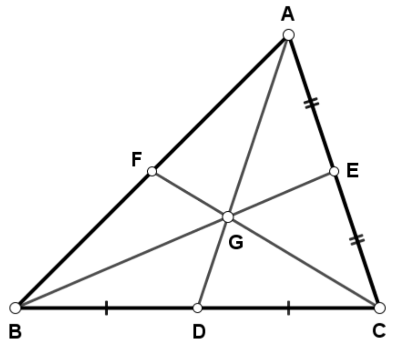
a.40 $Rr=\dfrac{abc}{4s}$

a.41 $R\cdot r_a=\dfrac{abc}{4\left(s-a\right)}$

a.42 $r\cdot r_a=\left(s-b\right)\left(s-c\right)$

a.43 $\dfrac{r_a}{r}=\dfrac{s}{s-a}$

a.44 $\frac{R}{r}=\frac{1}{4}\left(\frac{r_a}{r}-1\right)\left(\frac{r_b}{r}-1\right)\left(\frac{r_c}{r}-1\right)$



a.45 $r_a\cdot r_b+r_b\cdot r_c+r_c\cdot r_a=s^2$

a.46 $\sqrt{\frac{r_a\cdot r_b\cdot r_c}{r}}=s$

a.47 $\sqrt{\frac{r\cdot r_b\cdot r_c}{r_a}}+\sqrt{\frac{r\cdot r_c\cdot r_a}{r_b}}+\sqrt{\frac{r\cdot r_a\cdot r_b}{r_c}}=s$

a.48 $\frac{1}{r}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}$

a.49 $\frac{(r_a+r_b)(r_b+r_c)(r_c+r_a)}{r_a\cdot r_b+r_b\cdot r_c+r_c\cdot r_a}=4R$

a.50 $\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}=4\cdot\frac{r_a+r_b+r_c}{a+b+c}$

a.51 $r_a+r_b+r_c=4R+r$

a.52 $r_a^2+r_b^2+r_c^2=(r+4R)^2-2s^2$

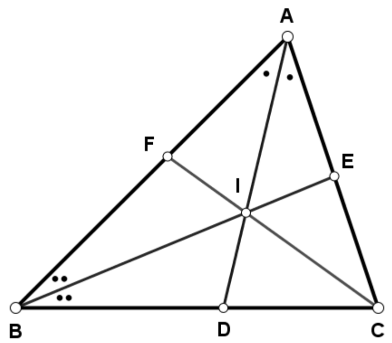
a.53 $a^2+b^2+c^2+r_a^2+r_b^2+r_c^2=16R^2-r^2$

a.54 $ab+bc+ca=4R\left(\frac{rr_a}{r_a-r}+\frac{rr_b}{r_b-r}+\frac{rr_c}{r_c-r}\right)$

a.55 $\frac{1}{r}=\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}$

a.56 $\frac{1}{r_a}=-\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}$

a.57 $\frac{1}{r}-\frac{1}{r_a}=\frac{2}{h_a}$



a.58 $\frac{h_a+h_b}{r_c}+\frac{h_b+h_c}{r_a}+\frac{h_c+h_a}{r_b}=6$

a.59 $\frac{R}{2r}=\frac{s^2}{h_ah_b+h_bh_c+h_ch_a}$

a.60 $2R=\frac{{w_a}^2}{h_a}\cdot\sqrt{\frac{{m_a}^2-{h_a}^2}{{w_a}^2-{h_a}^2}}$

a.61 $\frac{R}{r}=\frac{abc}{4(s-a)(s-b)(s-c)}$

a.62 $OI^2=R(R-2r)$

a.63 $OI_a^2 = R(R+2r_a)$

a.64 $OI^2 + OI_a^2 + OI_b^2 + OI_c^2 = 12R^2$

a.65 $3\left(OI^2 + OI_a^2 + OI_b^2 + OI_c^2\right) = 4\left(a^2+b^2+c^2\right) + 4\cdot OH^2$

a.66 $AI^2 = \frac{s-a}{s}\cdot bc$

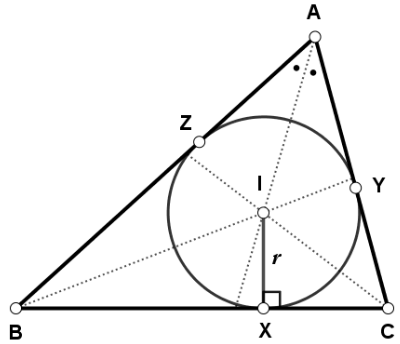
a.67 $AI_a^2 = \frac{s}{s-a}\cdot bc$

a.68 $AI \cdot AI_a = bc$

a.69 $\frac{CI\cdot CI_c}{BI\cdot BI_b} = \frac{b}{c}$

a.70 $AI^2 + BI^2 + CI^2 = ab + bc + ca - 12Rr$

a.71 $\frac{AI^2}{bc} + \frac{BI^2}{ca} + \frac{CI^2}{ab} = 1$



a.72 $\frac{ab}{CI_c^2} + \frac{bc}{AI_a^2} + \frac{ca}{BI_b^2} = 1$

a.73 $AI\cdot BI\cdot CI = 4Rr^2$

a.74 $AI_a\cdot BI_a\cdot CI_a = 4Rr_a^2$

a.75 $II_a\cdot II_b\cdot II_c = 16R^2 r$

a.76 $GI^2 + GI_a^2 + GI_b^2 + GI_c^2 = 16R^2 - \frac{4}{9}\left(a^2+b^2+c^2\right)$

a.77 $HI^2 = 4R^2 + 4Rr + 3r^2 - s^2$

a.78 $OH^2 = 9R^2 - \left(a^2+b^2+c^2\right)$

a.79 $GO^2 = R^2 - \frac{a^2+b^2+c^2}{9}$

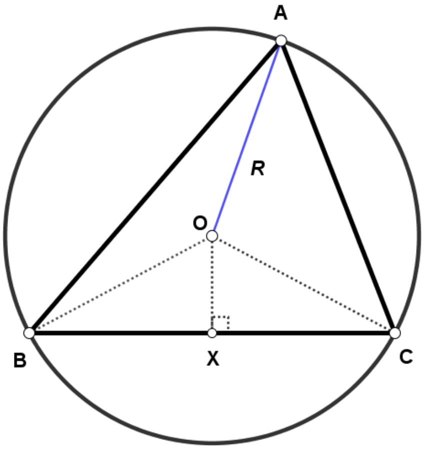
a.80 $[ABC] = \frac{1}{2}\cdot\sqrt[3]{abch_ah_bh_c}$

a.81 $[ABC] = \frac{1}{4}\left(a\cdot AH + b\cdot BH + c\cdot CH\right)$

a.82 $[ABC] = 2R^2\cdot\frac{h_a\cdot h_b\cdot h_c}{abc}$

a.83 $[ABC] = \sqrt{\frac{1}{2}R\cdot h_a\cdot h_b\cdot h_c}$

a.84 $[ABC] = \sqrt{r\cdot r_a\cdot r_b\cdot r_c}$



a.85	$[ABC]=\dfrac{r_a\cdot r_b\cdot r_c}{s}$
a.86	$[ABC]=a\cdot\dfrac{r_b\cdot r_c}{r_b+r_c}$
a.87	$[ABC]=(a+b)\cdot\dfrac{r\cdot r_c}{r+r_c}$
a.88	$[ABC]=\dfrac{r\cdot r_a(r_b+r_c)}{a}$
a.89	$[ABC]=r\cdot r_a\cdot\sqrt{\dfrac{4R-(r_a-r)}{r_a-r}}$
a.90	$[ABC]=r_a\cdot(s-a)$

Fig.2 – Basic geometric equalities

Activities and experimentation

As a first activity, the students practised recognizing possible correlations or dependencies between some given expressions and the ones illustrated in figure (Fig. 2). An example of possible expression given for this exercise can be found in figure (Fig. 3).

Proposal n. 2	
e.01	$a=2R\sin A$
e.02	$a=(b-c)\dfrac{r_b+r_c}{r_b-r_c}$
e.03	$\dfrac{a}{b}=\dfrac{(r_a-r)(r_b+r_c)}{r_ar_b+r_cr}$
e.04	$r_a=r\cdot\dfrac{s}{s-a}$
e.05	$\dfrac{a}{b}=\dfrac{r_ar_b+r_cr}{(r_b-r)(r_c+r_a)}$
e.06	$\dfrac{a-b}{c}=\dfrac{r_a-r_b}{r_a+r_b}$
e.07	$a^2+b^2+c^2=(r_a-r)(r_b+r_c)+(r_b-r)(r_c+r_a)+(r_c-r)(r_a+r_b)$
e.08	$abc=4r_aR(s-a)$
e.09	$R=\dfrac{1}{4}\cdot\sqrt{\dfrac{II_a\cdot II_b\cdot II_c}{r}}$

e.10	$r = \frac{r_a r_b r_c}{s^2}$
Identify any expressions that are consequences of one of the ninety basic geometric equalities. Indicate below their ID.	
Answers:	

Fig. 3 – recognition questionnaire

Subsequently, the students were asked to proof their ability to recognise expressions that combines two or more basic geometrical equalities between the ninety presented in figure (Fig. 2). The following table (Fig. 4) shows one of the boards used for this exercise.

Proposal n. 4	
f.01	$r = \frac{h_a h_b h_c}{h_a h_b + h_b h_c + h_c h_a}$
f.02	$\frac{AI_a \cdot BI_a \cdot CI_a}{R} = 4{r_a}^2$
f.03	$r^3 = \frac{abch_a h_b h_c}{4s^2\left(a+b+c\right)}$
f.04	$r_a r_b + r_b r_c + r_c r_a = s^2$
f.05	$\left[ABC\right] = \sqrt{s\left(s-a\right)\left(s-b\right)\left(s-c\right)}$
Identify any expressions that are a consequence of two or more of the ninety expressions provided. Indicate below their ID.	
Answers:	

Fig. 4 – Combination of basic geometric equalities

The students were asked to demonstrate one out of five proposed expression using their knowledge and the basic geometric equality in figure (Fig. 2). The following figure (Fig. 5) presents one of the tabs used for this group exercise.

Survey n. 1	
g.1	$\frac{b}{a} = \frac{r_a r_b + r_c r}{\left(r_a - r\right)\left(r_b + r_c\right)}$
g.2	$\frac{a+b}{a-b} = \frac{\left(s-c\right)\left(r+r_c\right)\left(r_a+r_b\right)}{cr\left(r_a - r_b\right)}$
g.3	$R = \frac{1}{4} \cdot \sqrt{\frac{II_a \cdot II_b \cdot II_c}{r_a r_b r_c}\left(r_a r_b + r_b r_c + r_c r_a\right)}$

g.4	$r = \frac{r_a r_b r_c}{s^2}$
g.5	$r_a = \frac{[ABC]}{(s-a)}$
Solve, in groups, one of the following geometric equalities. Basic geometric equalities can be used without demonstration.	
Resolution:	

Fig. 5 – Resolution, in groups, geometric equalities

Finally, figure (Fig. 6) presents one of the worksheets used, at the end of the experiment, to monitor the individual skills acquired.

Proposal n. 2	
h.1	$r = \frac{ah_a}{2s}$
h.2	$r = \frac{h_a h_b h_c}{h_a h_b + h_b h_c + h_c h_a}$
h.3	$[ABC] = 2R^2 \cdot \frac{h_a \cdot h_b \cdot h_c}{abc}$
Solve, individually, one of the following geometric equalities. Basic geometric equalities can be used without demonstration.	
Resolution:	

Fig. 6 – Final test. Resolution, individually, geometric equalities

Discussion and results

23 students, 16 males and 7 females, attending the eleventh grade participated in this research. Topics and in-depth analyses not usually taught during the normal course of study were dealt with.

Figure (Fig. 7) presents the percentages of exact answers reached by the students for the first exercise .

73.91% of the students have identified that the question e.4 is an immediate consequence of the basic geometric equality a.43 . This question was answered correctly by 81.25% of the 16 male students, which corresponds to 13 students and 57.14% of the female students which correspond to 4 people.

65.22% have identified that question e.8 is a consequence of equality a.41. This question was answered correctly by 68.75% of the males, which corresponds to 11 students and 57.14% of the students, or, in other words, to 4 people.

The percentage of right responses cannot be considered a good result. In fact, the task was only to observe and recognize eventual dependencies, using only the four fundamental operations of arithmetic. Also, the results of correct answers for the other cards were similar, in percentage, with the respect of the ones of the first exercise. The distribution range of right answer among different cards is ±1.7% .

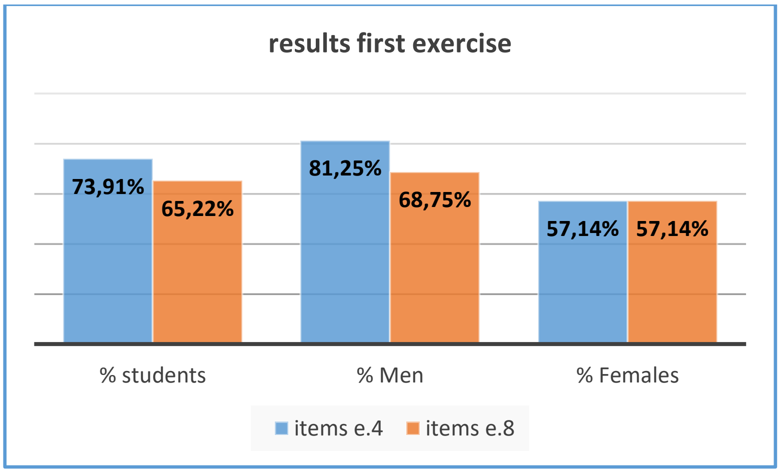


Fig. 7 – Results first exercise

The figure (Fig. 8) presents the percentages of the correct and non-correct answers for the second exercise.

The exercise asked to identify the equalities that would result combinations of two or more fundamental expressions.

8.70% of the students have identified that the question f.1 is a consequence of the equality a.55. Only two students answered incorrectly.

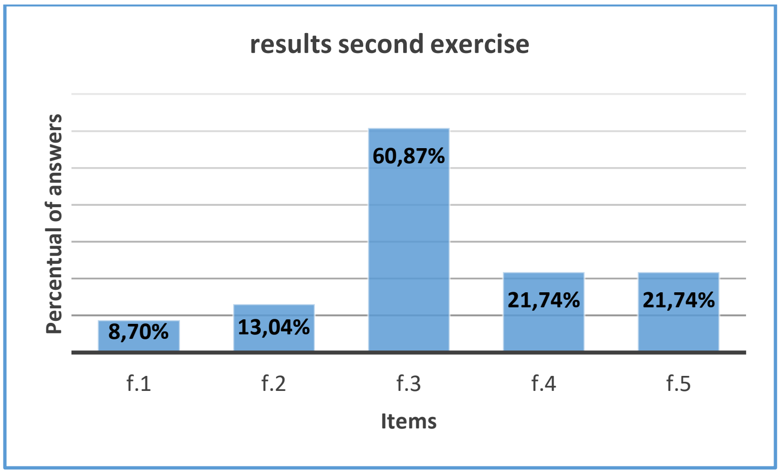


Fig. 8 – Results second exercise

In fact, the item is simply an equivalent formulation of the question a.55.

13.04% of the students have identified that the question f.2 is a consequence of the question a.74.

These (three) students answered incorrectly. In fact, item f.2 is an equivalent writing of question a.74.

60.87% of the students indicated that the question f.3 is a consequence of the questions a.3 and a.80. These (14) students answered exactly. In fact, the equality can be achieved by following simple algebraic operations using items a.3 and a.80.

21.74% of the students have identified that the question f.4 is a consequence of the question a.74. These five students answered incorrectly. In fact, item f.4 is identical to the question a.45.

21.74% of the 23 students have identified that the question f.5 is a consequence of the question a.4. These five students answered incorrectly. In fact, item f.5 is identical to question a.4.

We can think, after reading these results, that the students involved in this experiment, cannot read carefully, therefore, they cannot understand the deliveries of the tasks assigned to them.

This result can be further validated and extended (by mean of additional experiments) to a larger sample of students.

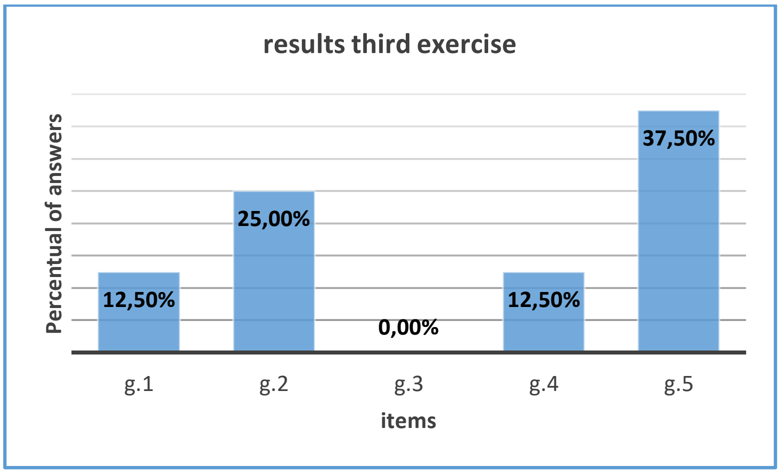


Fig. 9 – Results third exercise

Figure (figure 9) presents the percentages of the groups of students who have solved a specific item of the third exercise.

12.50% of the groups, which corresponds to 1 group out of eight, solved the question g.1.

25.00%, which corresponds to 2 groups, solved the question g.2. Questions g.1. and g.2. presents a medium difficulty.

The 00.00% solved the question g.3. The question presents a medium-high difficulty.

The 12.50%, has solved the question g.4. The question presents a medium difficulty.

37.50% which corresponds to 3 groups, solved the question g.5. The question presents a medium-low difficulty.

It is important to stress that all the groups have solved one of the three proposed questions successfully.

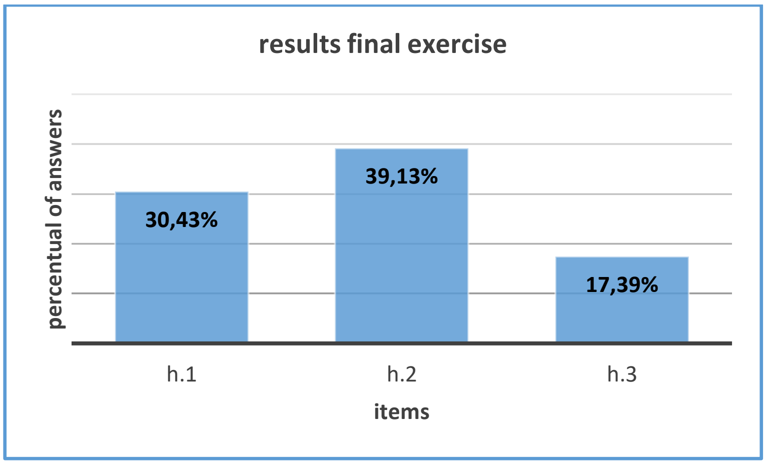


Fig. 10 – Results final exercise

As can be seen in the last figure (Fig. 10):

30.43% of the students, which corresponds to 7 people, solved the question h.1. The question does not present a high level of difficulty.

39.13% of the students, which corresponds to 9 people, solved the question h.2. The question does not present a high level of difficulty. This question was chosen by the largest number of students.

The 17.39% that corresponds to 4 students solved the question h.3. The question presents a higher level of difficulty with the respect to the other two proposed.

Finally, it should be noted that the 13.04% that corresponds to 3 students, has not been able to completely and/or correctly solve any item of those proposed.

The motivations of the choices made by the students were not asked during this study.

Conclusion

It is the intention of the authors to continue the experiment following the same basic methodology and gradually expanding the information provided to the students by adding angular elements and linear and angular elements together. This way of proceeding seems suitable to the authors in order to evaluate the ability of the students to manage information on increasingly complex problematic situations.

Declaration of Conflicting Interests

The author declared that they had no conflicts of interest concerning their authorship or the publication of this article.

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