Simulating Real Analysis-Honors Calculus in a sociocultural context

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Abstact *The sociocultural theory of learning has a tremendous clout in every branch of science, especially in mathematics .We try to reveal how a real analysis class experiences the benefits of a teaching approach based on the aforementioned learning theory. Predominant concepts that motivated us to do so are the fundamental Cauchy property and the Cesaro summability*

Key words *Vygotsky, sociocultural theory, Cesaro summability, Cauchy property*

Sommario *La teoria socioculturale dell'apprendimento ha un enorme influenza in ogni ramo della scienza, soprattutto in matematica. Cerchiamo di descrivere come una classe di analisi reale sperimenta i vantaggi di un approccio didattico basato sulla teoria dell'apprendimento di cui sopra. Concetti predominanti che ci hanno motivato a farlo sono la successione fondamentale di Cauchy e la sommabilita' di Cesaro.*

Parole chiave *Vygotsky, sociocultural theory, Cesaro summability, Cauchy property.*

Introduction

A theory that originated from the Soviet school of psychology has also a great clout on the western way of thinking. The sociocultural theory, inaugurated by Vygotsky, seems to eclipse political differences and unite people characterized by a common passion, i.e. mathematics education.

In the present article, the author simulates a real analysis-honors calculus course under the influence of the theory mentioned above. Since the author did not have the opportunity to teach in any of these courses before an audience so far, he resorted to combining comments on behalf of students regarding Real Analysis problems, through internet communication, personal contact, and general policies outside the classroom he really longs for. Later, he shaped that form of hypothetical-simulated class environment, trying to incorporate as many students' comments as possible and as many of his own reflections provided to them through the above mentioned communication means.

Methodology

The theory that we stick to is the sociocultural theory developed by Vygotsky (1978, 1986) and refined and exemplified by Davydov (1975b), Kozullin (1990). Learning takes place in high priority on the social level, and its individualized component follows that in time and importance. The learners are no passive subjects to be filled with knowledge. Learners have to be active participants. The presence of a more experienced, person can really enhance and deepen learners' knowledge, which is historically known as Zone of Proximal Development according to the pioneer of this theory.

The independence of the student is not ignored, but promoted, to make learners able to cope with harder and harder situations with the passing of time. Ergo, this independence is not considered as a beginning axiom, a prerequisite (Radford, 2008) but a corollary in our beloved mathematical terminology, a result.

The author collects data from internet communication and through personal contacts, even when the students are on their vacation on the wonderful island of Rhodes. On vacation, but still under the stress of successfully passing a Real Analysis-Honors Calculus course in their curriculum studies. Isn't it a real literally social approach? Even on holidays people tend to transfer with them what seems to trouble them, notwithstanding what vacations are for. These persons provide the author with motivation to deepen his insight while Real Analysis is taught and negotiate with them face to face or through e-mails, a better didactical contract in a broader gist of the word.

Thus, he posed them the problem that follows, an inevitable exercise while teaching this stuff, to sort of measure their ability to deal with it and to test his own sociocultural attitude in an effort to ameliorate any didactical misconcepts. *P* stands for the professor and the *S*'s for the students.

Main article

Definition the sequence a_n is Cesaro summable if the limit of its running averages

$$s_n = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

exists. Prove that every convergent sequence is Cesaro summable. Moreover, its running averages converge to the same limit (MacCluer, 2006).

Solution: Suppose $a_n \to a$. Since

$$\frac{a_1 + a_2 + \ldots + a_n}{n} - a = \frac{(a_1 - a) + (a_2 - a) + \ldots + (a_n - a)}{n}$$

it is enough to show the result when $a = 0$. Because the sequence a_n is convergent, it is bounded by a number, say *b*, i.e $|a_n| < b$ for all *n*. Let $\varepsilon > 0$ be given. Find a *p* such that $n > p$ implies $|a_n| < \frac{\varepsilon}{2}$. Ergo, for all $n > p$

$$\left|\frac{a_1 + a_2 + \ldots + a_n}{n}\right| = \left|\frac{a_1 + a_2 + \ldots + a_p}{n} + \frac{a_{p+1} + a_2 + \ldots + a_n}{n}\right| < \frac{pb}{n} + \frac{n-p}{n} \cdot \frac{\varepsilon}{2} << \frac{pb}{n} + \frac{\varepsilon}{2}$$

Then choose $q > \max\left(\frac{2pb}{\varepsilon}, p\right)$ so that for $n > q$

$$\frac{pb}{n} < \frac{\varepsilon}{2}$$

and the required result follows.

Since we are supposed to act, according to James Bank's terminology (Banks and McGee-Banks, 2010), we really interchange ourselves with the students in front of us. We are just emphasizing on the expression "it is enough to show the result with $a = 0$''. A limited number of students, if any, are able to understand in a fraction of our classroom time. Either we leave it as a homework problem, which will not be of much difficulty, or we solve it on the marker board.

P Whenever we get a sequence $b_n \to b$, we want to show that

$$\frac{b_1 + b_2 + \ldots + b_n}{n} \to b$$

or equivalently

$$\frac{b_1+b_2+...+b_n}{n}-b\to 0$$

I have already stated a formula, which dates back to earlier grades, manipulating algebraic expressions. After some time is left to the students, to work out that in their minds, someone shows up with the alacrity to write it on the blackboard.

S1 Since $b_n - b \to 0$, then it is running averages

$$\frac{(b_1-b)+(b_2-b)+...+(b_n-b)}{n}=\frac{b_1+b_2+...+b_n}{n}-b\to 0$$

i.e.

$$\frac{b_1+b_2+...+b_n}{n}\to b$$

P So, you do understand the importance that a specific number so easy to handle as 0, is enough to lead us to the solution of our initial problem.

S2 That can provide us with a method, that whenever we deal with a hard situation, we usually initiate a process involving 0, and then, we try to investigate if it covers all the cases.

S1 It is really fascinating to me that what we consider it -*it is really important how students tend to conceptualize ideas*- an unimportant number, I mean it does not play an important role in any arithmetic operation, but proves to be the cornerstone in a higher level mathematics course.

It would be critical here to open a new discussion about how students really attribute importance to such rudimentary concepts as numbers. But we contain ourselves again on the sociocultural aspect of the comment that an unimportant number is of paramount importance to a different situation, (Vygotsky 1978, 1986), Davydov (1975b). It would be ineludible to focus on the association between the course we teach and the social praxis we consider it is, by mentioning how many times society has taught us that nominal things in life come to play a fundamental role.

P You need to ask yourself what kind of person you really are. Are you the type who looks at all manner of solutions, or that sort of guy, who is usually looking for a shortcut? Are you accustomed to resorting to prescribed methods closely associated to the preceding material in earlier stages, in earlier grades or do you probably consider every new exercise as a new challenge?

The shortcut culture is the most widespread when it comes to problem solutions. But there still appears in our classroom a sort of dithering in which way to follow, habitually out of diffidence whether the short way will finally prove to be correct and finally, out of lack of the appropriate experience. An experience that the American oriented system of homework tends to cultivate an effort to deepen the acquisition of the mathematical knowledge. Notwithstanding that fact, it is really a social praxis, in the Vygotskian spirit, even if we are more experienced in our field than our students, at least we believe we are, to represent the horns of a dilemma that we also have encountered as students at certain times. Our culture of the past, no matter how much affected by the rapid incorporation of the technology, it still entails feelings. These feelings are really invariant, in the pure mathematical sense of the word, due to the unchanging human nature. Ergo, even if we are much older in age than our students, we seem to approach them by sharing their juvenile conceptualization. In a sociocultural environment (Cantoral, 2013), the factor age is not a separating one, but a reinforcement of the bond between teacher and student.

S2 You probably make it sound, especially when following the taught material from previous grades, as if Real Analysis were sort of continuation of that specific elementary school class.

P I sure do. I borrowed another colleague's expression *Calculus is what you learn in the fifth grade on an advanced scale*. My own adage would be, Honors Calculus too is what you learn in a previous grade even more advanced.

Here, without making a cause, we take advantage of the talk-producing atmosphere already generated, to instill into learners the pedagogical frame under whose influence we are working. It has been observed that, even we are not preparing future teachers, our students' audience really like that kind of deflection to a subject totally different from what they enrolled themselves in. For mathematically oriented students it is sort of an inspirational break, if they appear really exhausted by the mathematical jargon. For the rest of the students, it just provides them with a feeling that they participate in a class that would probably fit better their own personal inclinations and dexterities.

P Let me emphasize here the sociocultural ambient under whose repercussion we are moving. There is such a theory, going back to the Soviet era, and to a great mind, named Vygotsky (Vygotsky, 1978. 1986, Kozulin 1990), according to which the social and cultural component of knowledge takes place first. What you learn as individuals is secondary. I would be pleased if my attitude reflects that, in other words if communication and language are important factors in this course. You are familiar with what the word culture entails. It is built generation after generation, accomplishment after accomplishment, social activity after another social activity. Climbing up the ladder there is a rule that your social environment till now has already taught you. You need to step on the previous stair before you move to the next one. You may assign different names to the staircases whatsoever, but, no matter how you alternate among different names, the upper one is the continuation of the previous and so on. By adjusting the nomenclature, level 5 is the level (staircase) 2 on an advanced scale.

S2 To me is quite reassuring to hear that. It tries to embellish teaching obstacles encountered during our college life, especially when working hard on the assignment problems. Maybe what we deem as a great difficulty is eventually a simplicity we tend to forget, we tend to let it go unheeded.

P I would name it *latent simplicity*, a dormant one. Please pose the question to yourself. Try to awake experiences from your previous school years.

S2 I would rather say that resorting to the $\frac{\varepsilon}{2}$ technique is by far a fundamental technique. The textbook along with the instructor have made us understand how it is to reproduce, or maybe I can tell it in simple terms, rewrite the definition of the limit by replacing ε with $\frac{\varepsilon}{2}$.

P It surely is and I am glad that you recognized it. But allow me to deflect from that specific value and target our discussion we already started, to manipulate that to the extent that fits for our purpose. Am I somehow understood?

S3 I guess you mean all fractions of the kind $\frac{\varepsilon}{2}, \frac{\varepsilon}{3}$ …

S2 And not only fractions but also multiples…

P Yes. That freedom of choice that really facilitates and precipitates things. We have two primary goals, the right ε and the concomitantly right natural number associated with it. Would be that enough to make you solve more conveniently the above exercise?

By exchanging views with students, gives me an idea what their faces would look like. A certain degree of diffidence makes its appearance. As a devotee of socioculturalism, I try to analyze these demeanors. Either they are insecure to answer that or oppose it. I could have found during my teaching career another sort of relationship between the student and the teacher, but still the Vygotskian tool, language seems indispensable and strong enough while coping with the learning process as a whole. I do fell the necessity to hear and be heard. It is the reciprocity agreement, latent or not, that our didactical unsigned contract (Radford, 2008) induces me to comply with.

S2 The exercise presupposes our using the fact that since a sequence is convergent, it is bounded. Can I assume that such an association is highly likely to appear on an examination paper? You may clearly state it as a hint or are we supposed to move entirely on our own?

I feel I have to intervene with a comment once more. People keep complaining that our colleges are being converted into degree-grade awarding centers, whereas the real emphasis should have been placed on education itself. Naturally the ideal situation would be close to that belief, to our loyalty in the inerrancy of thinking and to our conviction that a university serves primarily the necessity to think or better improve how to think. But everything in our life is a social praxis, or let me paraphrase it a little, a combination, more mathematically, a permutation of praxis. Evaluation has been a driving force for all types of societies regardless the origin, the religion, the ethnic background. Students are exposed to that from an early age. They cannot avoid connecting what they learn with their evaluation process, even if this is viewed by some academics as a byproduct of knowledge. That is why I do not turn a deaf ear to these sort of questions, because it makes me really feel that I teach mathematics in a real and dynamic social realm, where evaluation and competition form basic components.

P Well that is a hard question for me too. I would say that as soon as you see the statement is convergent, you should have somewhere in your mind the three-fold concept either convergence-direct application of the definition, rewriting using your own expression, or convergence-boundedness, or convergence-Cauchy property. These are three interconnections that might really help you. As to how I am going to include it as an examination topic, you make me dither about what I am going to do. The hint you suggested would be an easy way to go through for both of us. Would I sound too mundane by referring to what you probably have heard lots of times before, solve as many problems as you can? Even if I make you feel jaded, try to discover by yourself similarities and differences between the assigned homework problems and the examined ones it will surely enhance your insight into viewing things more clearly and demystifying what used to cause you trouble.

The class seems quite skeptical about my telling them that. They probably need time to assimilate in both the constructivist and socioculturalist way, my words. There are moments like these which make me wish I could be capable of recording thoughts, cognitive pictures instead of voices. Who knows, technology advances so rapidly that my dream might materialize sooner than expected.

Back again to our analyzing process

S3 The clearest part is the work with the absolute value and the inequalities -*there is usually a preceding course in the American colleges, Applied Calculus or something that sufficiently prepares them for what ensues*-. The evil thing -*I really relish their using their own jargon, the sentimental dimension of our lecture to the boon of learning*- is the value q. In my opinion, it could have been a bonus question. It is surely the imaginative mind, the mind that adores mathematics, how else I can really name it…

P It is the culmination of the imagination, if you just look at it without the appropriate wording accompanied. If I made it a separate exercise, let me say a new one, I have the hunch it would open a way for your being capable of manipulating that easier

S3 What exactly you mean by that

P Solve the following exercise. Find the natural number *n* such that

$$\frac{pb}{n}+\frac{\varepsilon}{2}<\varepsilon$$

S3 Ninth grade problems would increase enrollment

P That is what I am looking for

Some moments elapse

P High school class, isn't it?

I get the impression that they agree on that.

P Now that evil thing that you mentioned before has somehow been sanctified?.

The class atmosphere is pretty reassuring and time will surely tell.

Conclusion

- Remind the students what they have learned before, and let them ask themselves how well they have mastered the material taught so far. Everybody today talks about a multi-component evaluation, evaluation by peers, evaluations by our superiors and inferiors, evaluation by ourselves. Self-evaluation is really a hard part, in the way we discussed it above we motivate students to evaluate themselves on a concrete basis the so far accumulated knowledge and experience. They come closer not only to their own difficulties but also to the difficulty in the teacher's job to properly and unbiased evaluate them. Moreover, a lot of times, even at the college level, our vocational realm, where a certain level of knowledge and mathematical awareness is presupposed, we encounter situations, especially while grading-evaluation emanates naturally- their homework and exams that cause us embarrassment in varying degrees of intensity. The usual colleagues' comment, which surely does not exculpate me in any way, is to succumb to the *generation gap trap*. The older the generation, the better prepared it is, or at least it though it was. This the easy way to obviate dealing with guilty for our own teaching underperformance. Have we ever asked ourselves how easy it is to leave behind anything, even fundamental, as we keep expanding the knowledge, the information input that our minds and by the same token our student's mind receive? Is it so cumbersome to get them back to the grades, where basic stuff was covered, and refresh their memory? It surely bolsters their moral and puts aside their mood for retreat, partial or full.
- Crack the problem into pieces, according to the difficulties presented or anticipated. A pure constructivist would renounce that even as a concept, it would restrict according to him the student's independence, which should not be violated anyway. On the other hand, a socioculturalist would rather deem it important in the direction of accomplishing her/his goal. We do intervene where the student feels that he/she cannot proceed by himself/herself, literally violate his/her learning independence, autonomy. But we create an ambient dynamically, gradually constructive. In a similar exercise, the students will be more adequately prepared to deal with it, they will have increased their robustness to the nature of difficulties. The kind of independence that the student will experience from now on is an on-the-go process, a conquered independence. Our whole demeanor reflects another socioculturalist lodestone that autonomy, independence on behalf of the student emerges as a result, it is not presupposed. Let us not forget that in the same way our teaching independence is significantly reinforced on the basis of reciprocity, a clear social and cultural praxis.

Feelings, a product and at the same time an irrefutable catalyst in every social process, are apparent and compatible with pure logic. Knowledge by itself does not suffice to transform any lecture into a success. When we ask students to elaborate on their difficulties and conceptual impediments, our behavior evinces how we care about them. On the other hand, if they feel free to discuss their problems with us, they are beneficial to implement our teaching duties, in other words they care about us too in a broader sense. Whether they know it or not, they assign value to our existence in the classroom, which in my opinion sometime in the middle of the semester is totally heeded and expressed toward us. James Bank's three popular verbs (Banks and McGee-Banks, 2010) are employed once more: know, care and act. The first two were described above. About the third one it is really straightforward in my opinion. Being there for our students anytime they need us involves our whole dedication. It means leaving aside all manners of personal adversities, misfortunes, strokes of life. The same is anticipated on their side. Ergo, we have been acting even whether this remains dormant in our deeper consciousness. Let us make an effort and grade our acting performance on a weekly or on a daily basis, if possible. It will reveal us another side of ourselves that really induced us to wish to become teachers, and if we were given a second chance in life, no matter the usual complaints that are heard over the attack on education, we would surely follow the same choice.

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